METHODS TO DETERMINE COXETERPOLYNOMIALS

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Abstract. A reduction formula for the characteristic polynomials ϕ_Λ of the Coxetermatrices of certain split finite-dimensional algebras Λ is proved. In the hereditary case, this yields explicit expressions for the Coxeterpolynomials of large families of quivers. Moreover, a combinatorial interpretation of the entries of the Coxetermatrices of path algebras gives formulas for Coxeterpolynomials of some quivers which cannot be treated by the above mentioned reduction process.

1. Introduction

The purpose of this paper is to establish a reduction formula for the characteristic polynomial ϕ_Λ of the Coxetermatrix of a split finite-dimensional algebra Λ. In fact, when Λ is put together from subalgebras in a certain natural fashion, we express ϕ_Λ in terms of the Coxeterpolynomials of these subalgebras. In concrete computations, repeated application of this reduction principle offers a significant edge over direct use of the definition. In the hereditary case, in particular, this principle yields explicit expressions for the Coxeterpolynomials of large families of quivers. Moreover, a combinatorial interpretation of the entries of the Coxetermatrices of hereditary path algebras allows us to establish formulas for Coxeterpolynomials of some quivers which cannot be treated by the above mentioned reduction process.

The significance of the Coxetermatrix Φ_Λ and the Coxeterpolynomial ϕ_Λ of a finite-dimensional algebra Λ lies in the following observation [4]: Namely, if Λ has finite global dimension, then the derived category $\mathrm{D}^b(\Lambda)$ has Auslander-Reiten triangles, and the resulting Auslander-Reiten translation yields an endomorphism with matrix Φ_Λ on the level of the Grothendieck group $\mathrm{G}_0(\mathrm{D}^b(\Lambda)) = \mathrm{G}_0(\Lambda)$. As a consequence, the Coxeterpolynomial ϕ_Λ – its zeroset, in particular – contains valuable information on the growth behaviour of iterated Auslander-Reiten translates.

Throughout, Γ will be a finite quiver with vertex set $V\Gamma$ and $\Lambda = K\Gamma/I$ will be a path algebra modulo an ideal of relations over a field K such that $\dim_K \Lambda < \infty$ (see e. g. [5], sec. 2.1, for the definitions, but note that we compose paths like maps: if p

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is path from i to j and q is a path from j to k, then qp denotes the composite path from i to k). We will denote the primitive idempotent of Λ corresponding to $i \in \mathrm{V}\Gamma$ with e_i. Recall that the $\mathrm{V}\Gamma \times \mathrm{V}\Gamma$ matrix

$$C_\Lambda = (\dim_K e_i \Lambda e_j)_{(i,j)\in \mathrm{V}\Gamma\times\mathrm{V}\Gamma}$$

is called the Cartan matrix of Λ and that, in case $|C_\Lambda| = \det(C_\Lambda) \neq 0$ (which is always satisfied if Λ has finite global dimension), the Coxetermatrix of Λ is defined as

$$\Phi_\Lambda := -{}^tC_\Lambda C_\Lambda^{-1},$$

where ${}^tC_\Lambda$ denotes the transpose of the matrix C_Λ. We will study the Coxeterpolynomial $\phi_\Lambda(T) = |TE - \Phi_\Lambda|$ of Λ.

Our main result deals with the situation where $\Lambda = K\Gamma/I$, and Γ is the union of two quivers Γ_1 and Γ_2 which share only a single vertex r such that I can be generated by relations which involve no paths properly passing through r. If Λ_i, resp. $\check{\Lambda}_i$, denotes the algebra obtained from $K\Gamma_i$, resp. $K(\Gamma_i \setminus \{r\})$, by factoring out the obvious contraction of I, we obtain

$$\phi_\Lambda = \phi_{\Lambda_1}\phi_{\check{\Lambda}_2} + \phi_{\check{\Lambda}_1}\phi_{\Lambda_2} - (T+1)\phi_{\check{\Lambda}_1}\phi_{\check{\Lambda}_2},$$

whenever $|C_\Lambda| \neq 0$.

Based on this equation, we have developed a program using the computer algebra system 'Maple', which is capable of symbolically generating formulas for the Coxeterpolynomials of large classes of path algebras, as well as of efficiently computing Coxeterpolynomials of concretely given path algebras. The detailed discussion of this program and its complexity is contained in [2]. The program is called 'coxpoly' and is freely available [3].

2. The main result

Let r be a vertex of the quiver Γ and p a path in Γ. We say that p *properly passes through* r, if p can be written in the form $p = p_2 e_r p_1$ with paths p_1, p_2 in Γ of length ≥ 1.

For $n \in \mathbb{N}_0$, we say that p *properly passes through* r *precisely* n *times*, if p may be written in the form $p = p_{n+1} e_r p_n e_r \cdots e_r p_1$ with paths $p_1, \ldots, p_{n+1}$ of length ≥ 1 which do not properly pass through r.

Moreover, an ideal I of relations in $K\Gamma$ is called r*-separated*, in case I can be generated as an ideal by a set R of relations such that for every $\sum_j \mu_j w_j \in R$ with $\mu_j \in K \setminus \{0\}$ and distinct paths w_j in Γ, none of the w_j properly passes through r.

We denote by $\Gamma \setminus \{r\}$ the quiver obtained from Γ by deleting the vertex r and all adjacent arrows. If Γ is the empty quiver without vertices and arrows, then $K\Gamma$ is the trivial zero-dimensional K-algebra with Coxeterpolynomial 1.

The conclusion of the following Lemma essentially allows us to count nonzero residue classes of paths in a similar way as we count paths in the case of finite-dimensional path algebras without relations:

Lemma 2.1. *Consider a finite-dimensional K-algebra $\Lambda = K\Gamma/I$ and let $r \in \mathrm{V}\Gamma$ be a vertex such that I is r-separated. Then*

(a) $$\dim_K e_r \Lambda e_r = 1$$

(b) *Set $\check{\Gamma} := \Gamma \setminus \{r\}$ and $\check{\Lambda} := K\check{\Gamma}/(I \cap K\check{\Gamma})$. The assignment*

$$u \otimes v \oplus w \mapsto uv + w$$

yields an isomorphism

$$\Lambda e_r \underset{K}{\otimes} e_r \Lambda \ \oplus \ \check{\Lambda} \xrightarrow{\sim} \Lambda$$

of $\check{\Lambda}$-$\check{\Lambda}$-bimodules.

Proof. We denote by $P^{(n)}$ the K-subspace of $K\Gamma$ generated by all paths starting and ending in r and properly passing through r precisely $n-1$ times. Let $R \subset I$ be a generating set of relations which do not involve paths properly passing through r.

As an immediate consequence of the definitions, we get: If ρ is an element of R, p is a path starting in r and q is a path ending in r, then

$$q\rho p \in \bigcup_{n \geq 1} P^{(n)}. \tag{1}$$

Hence,

$$e_r I e_r = \bigoplus_{n \geq 1} I \cap P^{(n)}. \tag{2}$$

We write $\bar{P}^{(n)} := P^{(n)}/(I \cap P^{(n)})$. If moreover we denote by J the Jacobson radical of Λ, i. e. the canonical image modulo I of the ideal of $K\Gamma$ generated by all arrows, equation (2) yields

$$e_r J e_r = \bigoplus_{n \geq 1} \bar{P}^{(n)}. \tag{3}$$

Next, we prove

$$\bar{P}^{(n)} \simeq \bigotimes^n \bar{P}^{(1)},$$

where $\bigotimes^n \bar{P}^{(1)}$ is the n-fold tensor product of $\bar{P}^{(1)}$ with itself, taken over K. Together with $\dim_K \Lambda < \infty$ and (3), this will give us $\bar{P}^{(1)} = 0$ and hence (a).

The exact sequence

$$I \cap P^{(1)} \to P^{(1)} \to \bar{P}^{(1)} \to 0$$

induces the upper row of the following commutative diagram with exact rows:

$$\begin{tikzcd}
\bigoplus_{k=1}^{n} \left(\bigotimes^{k-1} P^{(1)} \otimes (I \cap P^{(1)}) \otimes \bigotimes^{n-k} P^{(1)} \right) \arrow[r] \arrow[d, "f"] & \bigotimes^{n} P^{(1)} \arrow[r] \arrow[d, "g"', "\sim"] & \bigotimes^{n} \bar{P}^{(1)} \arrow[r] \arrow[d, "h"] & 0 \\
I \cap P^{(n)} \arrow[r] & P^{(n)} \arrow[r] & \bar{P}^{(n)} \arrow[r] & 0
\end{tikzcd}$$

Here, the maps f and g are defined by

$$g(p_1 \otimes p_2 \otimes \cdots \otimes p_n) = p_1 p_2 \cdots p_n$$

and

$$f(p_1 \otimes \cdots \otimes p_{k-1} \otimes x \otimes p_{k+1} \otimes \cdots \otimes p_n) = p_1 \cdots p_{k-1} x p_{k+1} \cdots p_n.$$

In order to prove that h is an isomorphism, it suffices to show that f is onto since g is an isomorphism. We only have to consider the case $n \geq 2$. Pick $x \in I \cap P^{(n)}$ and write $x = \sum_i \lambda_i q_i \rho_i p_i$ with $\lambda_i \in K \setminus \{0\}$, $\rho_i \in R$ and paths p_i, q_i starting resp. ending in r. Because of (1) and (2), we may assume $q_i \rho_i p_i \in P^{(n)}$ for all i. Write $p_i = p_{i2} e_r p_{i1}$ and $q_i = q_{i2} e_r q_{i1}$ where p_{i2} and q_{i1} have smallest possible length ≥ 0. Then we have $q_{i1} \rho_i p_{i2} \in I \cap P^{(1)}$. Moreover either $p_{i1} = e_r$ and $q_{i2} \in P^{(n-1)}$, or $p_{i1} \in P^{(n-1)}$ and $q_{i2} = e_r$, or else there is some $k_i \in \{2, \ldots, n-1\}$ such that $p_{i1} \in P^{(n-k_i)}$ and $q_{i2} \in P^{(k_i - 1)}$. In either case, we get $q_i \rho_i p_i \in \operatorname{Im} f$ since multiplication of paths yields an isomorphism $\bigotimes^k P^{(1)} \xrightarrow{\sim} P^{(k)}$. Consequently, $x \in \operatorname{Im} f$.

Now consider the map $\phi : \Lambda e_r \otimes_K e_r \Lambda \oplus \check{\Lambda} \to \Lambda$ in part (b) of the Lemma. Obviously, it is well-defined, $\check{\Lambda}$-$\check{\Lambda}$-bilinear and surjective. In order to find a left inverse ψ to ϕ, we start with a K-linear map $\psi_0 : K\Gamma \to \Lambda e_r \otimes_K e_r \Lambda \, \oplus \, \check{\Lambda}$, defined on the paths p in Γ as follows: if p can be written in the form $p = p_2 e_r p_1$ with paths p_1, p_2 (not necessarily of length ≥ 1), we set

$$\psi_0(p) := (p_2 + I e_r) \otimes (p_1 + e_r I) \, \oplus \, 0.$$

This is well-defined, because if $p = q_2 e_r q_1$ is a different factorization of this kind, then either $q_1 = x q_1'$ or $q_2 = q_2' x$ with a suitable $x \in V^{(1)}$. But $V^{(1)} \subset I$ in view of (a), and thus $(q_2 + I e_r) \otimes (q_1 + e_r I) = 0$. Analogously, one derives $(p_2 + I e_r) \otimes (p_1 + e_r I) = 0$. If p cannot be written in the form $p_2 e_r p_1$, we set

$$\psi_0(p) := 0 \oplus (p + I \cap K\check{\Gamma}).$$

Now suppose $x = \sum_i \lambda_i q_i \rho_i p_i \in I$. If p_i can be written in the form $p_i = p_{i2} e_r p_{i1}$ with paths p_{i1} and p_{i2}, then $q_i \rho_i p_{i2} \in I e_r$ and $\psi_0(q_i \rho_i p_i) = 0$. Similarly, if q_i admits a factorization $q_i = q_{i2} e_r q_{i1}$, we have $\psi_0(q_i \rho_i p_i) = 0$. The remaining case is $q_i \rho_i p_i \in$

$I \cap K\check{\Gamma}$, and again we obtain $\psi_0(q_i\rho_i p_i) = 0$. Thus $\psi_0(x) = 0$ and ψ_0 induces a K-linear map $\psi : \Lambda \to \Lambda e_r \otimes_K e_r \Lambda \oplus \check{\Lambda}$ which by construction is left inverse to ϕ. $\square$

The *union* of quivers is given by the union of the vertex sets and the disjoint union of the arrow sets.

Now we are in a position to prove the main result:

Theorem 2.2. *Let Γ_1, Γ_2 be two finite quivers with $\mathrm{V}\Gamma_1 \cap \mathrm{V}\Gamma_2 = \{r\}$, and let Γ be the union of Γ_1 and Γ_2. Suppose that $I \subset K\Gamma$ is an r-separated ideal of relations such that $\Lambda := K\Gamma/I$ is finite-dimensional. Set $\check{\Gamma}_1 := \Gamma_1 \setminus \{r\}$ and $\check{\Gamma}_2 := \Gamma_2 \setminus \{r\}$ and define the algebras $\Lambda_1, \check{\Lambda}_1, \Lambda_2, \check{\Lambda}_2$ canonically:*

$$\Lambda_i := K\Gamma_i/(I \cap K\Gamma_i) \quad \textit{and} \quad \check{\Lambda}_i := K\check{\Gamma}_i/(I \cap K\check{\Gamma}_i) \quad \textit{for } i = 1, 2.$$

Then

$$|C_{\Lambda_1}| = |C_{\check{\Lambda}_1}|, \quad |C_{\Lambda_2}| = |C_{\check{\Lambda}_2}| \quad \textit{and} \quad |C_\Lambda| = |C_{\Lambda_1}||C_{\Lambda_2}|.$$

If this last determinant is nonzero, the Coxeterpolynomial of Λ is

$$\phi_\Lambda = \phi_{\Lambda_1}\phi_{\check{\Lambda}_2} + \phi_{\check{\Lambda}_1}\phi_{\Lambda_2} - (T+1)\phi_{\check{\Lambda}_1}\phi_{\check{\Lambda}_2}.$$

Proof. We need some additional notation: for every $i \in \mathrm{V}\check{\Lambda}_1$, let $a_i := \dim_K e_r\Lambda e_i$ and $\tilde{a}_i := \dim_K e_i\Lambda e_r$. Accordingly, for every $i \in \mathrm{V}\check{\Lambda}_2$, set $b_i := \dim_K e_r\Lambda e_i$ and $\tilde{b}_i := \dim_K e_i\Lambda e_r$. We consider a, $\tilde{a}$, b and $\tilde{b}$ as column vectors and write C, C_1, C_2, $\check{C}_1$, $\check{C}_2$ instead of C_Λ, C_{Λ_1}, C_{Λ_2}, $C_{\check{\Lambda}_1}$, $C_{\check{\Lambda}_2}$.

First we observe that $e_r\Lambda_1 e_i = e_r\Lambda e_i$ and $e_j\Lambda_1 e_r = e_j\Lambda e_r$ since there are no arrows connecting $\mathrm{V}\check{\Gamma}_1$ and $\mathrm{V}\check{\Gamma}_2$ and $\dim_K e_r\Lambda e_r = 1$ by Lemma 2.1. Moreover, $I \cap K\Gamma_1$ is an r-separated ideal in $K\Gamma_1$ because every relation which does not involve any paths properly passing through r lies either in $K\Gamma_1$ or in $K\Gamma_2$. Applying Lemma 2.1 to Λ_1, we see that:

$$\dim_K e_r\Lambda_1 e_r = 1 \quad \text{and}$$
$$\dim_K e_i\Lambda_1 e_j = \dim_K e_i\check{\Lambda}_1 e_j + \tilde{a}_i a_j \quad \text{for all } i, j \in \mathrm{V}\check{\Gamma}_1.$$

Thus

$$C_1 = \left(\begin{array}{c|c} \check{C}_1 + \tilde{a}\,{}^t a & \tilde{a} \\ \hline {}^t a & 1 \end{array}\right),$$

and by subtracting suitable multiples of the last row from the others, we get $|C_1| = |\check{C}_1|$. Analogously, we obtain $|C_2| = |\check{C}_2|$. If we set $\check{\Gamma} := \Gamma \setminus \{r\}$ and $\check{\Lambda} := K\check{\Gamma}/(I \cap K\check{\Gamma})$, another application of Lemma 2.1 together with

$$C_{\check{\Lambda}} = \left(\begin{array}{c|c} \check{C}_1 & 0 \\ \hline 0 & \check{C}_2 \end{array}\right)$$

gives us

$$C = \left(\begin{array}{c|c|c} \check{C}_1 + \tilde{a}\,{}^t\!a & \tilde{a} & \tilde{a}\,{}^t b \\ \hline {}^t\!a & 1 & {}^t b \\ \hline \tilde{b}\,{}^t\!a & \tilde{b} & \check{C}_2 + \tilde{b}\,{}^t b \end{array}\right),$$

and hence $|C| = |\check{C}_1||\check{C}_2|$.

Now suppose C is invertible over $\mathbb{Q}$. Then the same is true for C_1, C_2, $\check{C}_1$ and $\check{C}_2$, and we write Φ, Φ_1, Φ_2, $\check{\Phi}_1$, $\check{\Phi}_2$ instead of Φ_Λ, Φ_{Λ_1}, Φ_{Λ_2}, $\Phi_{\check{\Lambda}_1}$, $\Phi_{\check{\Lambda}_2}$.

If A and B are invertible matrices such that $B = SA\,{}^tS$ for some invertible matrix S, we will write $A \sim B$. Note that in this case $S(-{}^tAA^{-1})S^{-1} = -{}^tBB^{-1}$, and therefore $-{}^tAA^{-1}$ and $-{}^tBB^{-1}$ have the same characteristic polynomial.

Obviously, we have

$$C_1 = \left(\begin{array}{c|c} \check{C}_1 + \tilde{a}\,{}^t\!a & \tilde{a} \\ \hline {}^t\!a & 1 \end{array}\right) \sim \left(\begin{array}{c|c} \check{C}_1 & \tilde{a} - a \\ \hline 0 & 1 \end{array}\right) =: D_1.$$

Moreover, observe that

$$D_1^{-1} = \left(\begin{array}{c|c} \check{C}_1^{-1} & \check{C}_1^{-1}(a - \tilde{a}) \\ \hline 0 & 1 \end{array}\right),$$

and hence

$$-{}^tD_1 D_1^{-1} = \left(\begin{array}{c|c} \check{\Phi}_1 & \check{\Phi}_1(a - \tilde{a}) \\ \hline {}^t(a - \tilde{a})\check{C}_1^{-1} & {}^t(a - \tilde{a})\check{C}_1^{-1}(a - \tilde{a}) - 1 \end{array}\right).$$

Similarly, Φ_2 and

$$\left(\begin{array}{c|c} {}^t(b - \tilde{b})\check{C}_2^{-1}(b - \tilde{b}) - 1 & {}^t(b - \tilde{b})\check{C}_2^{-1} \\ \hline \check{\Phi}_2(b - \tilde{b}) & \check{\Phi}_2 \end{array}\right)$$

have the same characteristic polynomial. Applying the same reasoning to the full algebra Λ and using

$$\Phi_{\check{\Lambda}} = \left(\begin{array}{c|c} \check{\Phi}_1 & 0 \\ \hline 0 & \check{\Phi}_2 \end{array}\right),$$

we obtain that Φ and

$$\left(\begin{array}{c|c|c} \check{\Phi}_1 & \check{\Phi}_1(a - \tilde{a}) & 0 \\ \hline {}^t(a - \tilde{a})\check{C}_1^{-1} & \lambda - 1 & {}^t(b - \tilde{b})\check{C}_2^{-1} \\ \hline 0 & \check{\Phi}_2(b - \tilde{b}) & \check{\Phi}_2 \end{array}\right)$$

have the same characteristic polynomial as well. Here, we set

$$\lambda = {}^t(a-\tilde{a})\check{C}_1^{-1}(a-\tilde{a}) + {}^t(b-\tilde{b})\check{C}_2^{-1}(b-\tilde{b}).$$

(Note that the quadratic form $\chi(x) = {}^t x C_{\check{\Lambda}}^{-1} x$ has significance in its own right because it is tightly connected to the Euler characteristic of the algebra $\check{\Lambda}$, see [5], p. 70.)
If finally we abbreviate

$$\begin{aligned}
\alpha &= -(T+1),\\
\alpha_1 &= (T+1) - {}^t(a-\tilde{a})\check{C}_1^{-1}(a-\tilde{a}),\\
\alpha_2 &= (T+1) - {}^t(b-\tilde{b})\check{C}_2^{-1}(b-\tilde{b}),
\end{aligned}$$

we recognize the theorem as a consequence of the following Lemma. $\square$

Lemma 2.3. *Let R be a commutative ring and $F \in \mathrm{M}_n(R)$ a matrix of the following form:*

$$F = \left(\begin{array}{c|c|c} F_1 & f_1 & 0 \\ \hline g_1 & \alpha_1+\alpha+\alpha_2 & g_2 \\ \hline 0 & f_2 & F_2 \end{array}\right)$$

where $F_1 \in \mathrm{M}_{n_1}(R)$, $F_2 \in \mathrm{M}_{n_2}(R)$, $n_1+n_2+1 = n$, $\alpha, \alpha_1, \alpha_2 \in R$, $f_1, {}^t g_1 \in R^{n_1}$ and $f_2, {}^t g_2 \in R^{n_2}$. Then

$$|F| = \left|\begin{array}{c|c} F_1 & f_1 \\ \hline g_1 & \alpha_1 \end{array}\right| |F_2| + |F_1| \left|\begin{array}{c|c} \alpha_2 & g_2 \\ \hline f_2 & F_2 \end{array}\right| + \alpha|F_1||F_2|.$$

Proof. Develop the determinant with respect to the (n_1+1)-th row or column. $\square$

An obvious induction yields the following generalization of Theorem 2.2:

Corollary 2.4. *Let Γ_i, $i = 1,\dots,t$, be finite quivers with $\mathrm{V}\Gamma_i \cap \mathrm{V}\Gamma_j = \{r\}$ for $i \neq j$, and let Γ be the union of the Γ_i. Suppose that $I \subset K\Gamma$ is an r-separated ideal of relations such that $\Lambda := K\Gamma/I$ is finite-dimensional. Set $\Lambda_i := K\Gamma_i/(I \cap K\Gamma_i)$ and $\check{\Gamma}_i := \Gamma_i \setminus \{r\}$ and $\check{\Lambda}_i := K\check{\Gamma}_i/(I \cap K\check{\Gamma}_i)$ for $i = 1,\dots,t$. Then*

$$|C_{\Lambda_i}| = |C_{\check{\Lambda}_i}| \quad \textit{for all } i \quad \textit{and} \quad |C_\Lambda| = \prod_{i=1}^{t} |C_{\Lambda_i}|,$$

and if this last determinant is nonzero, we have

$$\phi_\Lambda = \Big(\prod_{i=1}^{t} \phi_{\check{\Lambda}_i}\Big)\Big(\sum_{i=1}^{t} \frac{\phi_{\Lambda_i}}{\phi_{\check{\Lambda}_i}} - (t-1)(T+1)\Big). \quad \square$$

3. The hereditary case

If Λ is hereditary, i. e. if $\Lambda = K\Gamma$ and Γ is a finite quiver without oriented cycles, then the matrix C_Λ, and consequently also Φ_Λ and ϕ_Λ, depend only on the quiver Γ and not on the base field K. In fact,

$$C_\Gamma := C_\Lambda = (\#\text{ paths from } j \text{ to } i \text{ in } \Gamma)_{(i,j)\in V\Gamma\times V\Gamma},$$

and the adjacency matrix of Γ,

$$A_\Gamma := (\#\text{ arrows from } j \text{ to } i \text{ in } \Gamma)_{(i,j)\in V\Gamma\times V\Gamma},$$

satisfies $C_\Gamma^{-1} = E - A_\Gamma$ where E is the $V\Gamma\times V\Gamma$ identity matrix. With this in mind, one obtains a combinatorial interpretation of the entries of $\Phi_\Gamma := \Phi_\Lambda$ as follows. Namely, for $i, j \in V\Gamma$, a *twisted path from i to j* is defined to be a sequence $(\beta, \alpha_{n-1}, \dots, \alpha_1)$ of arrows in Γ such that $(\alpha_{n-1}, \dots, \alpha_1)$ is a path from i to some vertex k and β is an arrow from j to k. Roughly speaking, a twisted path consists of a 'regular' path to which we attach an inverted arrow. With this convention, we obtain:

Proposition 3.1. *The Coxetermatrix of a finite quiver Γ without oriented cycles is*

$$\Phi_\Gamma = (\#\textit{ twisted paths from } i \textit{ to } j - \#\textit{ paths from } i \textit{ to } j)_{(i,j)\in V\Gamma\times V\Gamma}.$$

Proof. This follows from the above discriptions of C_Γ and its inverse and the fact that the number of twisted paths from i to j is the sum of all products (number of paths from i to k)$\times$(number of arrows from j to k) for $k \in V\Gamma$. $\square$

It is interesting to note that the reduction formulas for the Coxeterpolynomial and for the characteristic polynomial of the adjacency matrix for quivers of the type considered in Corollary 2.4 are exactly the same. (Of course, the term $(T+1)$, which is the Coxeterpolynomial of a one-point quiver without arrows, has to be replaced by the corresponding characteristic polynomial of the adjacency matrix, i. e. by T.) The reason can again be found in Lemma 2.3.

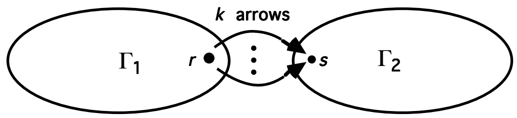
Observe moreover that there is a tight connection between ϕ_Γ and the characteristic polynomial of the underlying undirected graph in case every vertex of Γ is either a sink or a source; see e. g. [1].

We set

$$v_k := \frac{T^k - 1}{T - 1} \quad \text{for every } k \in \mathbb{Z}.$$

The linear graph A_k with $k \geq 0$ vertices has Coxeterpolynomial v_{k+1} as one easily derives from Theorem 2.2 by induction. The orientation of the arrows does not have any impact on the formula here; indeed, this is obviously true for A_2, and thus follows for higher values of k. In view of these remarks, a straightforward computation yields the following

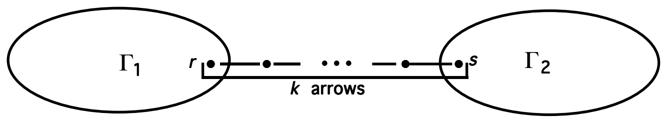
Corollary 3.2. *Let Γ_1 and Γ_2 be finite quivers without oriented cycles. Then the quiver*



has Coxeterpolynomial

$$\phi_{\Gamma_1}\phi_{\Gamma_2} - k^2 T \phi_{\Gamma_1\setminus\{r\}}\phi_{\Gamma_2\setminus\{s\}}.$$

The quiver



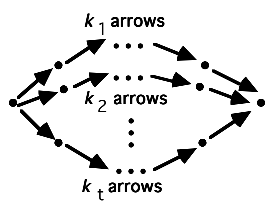
has Coxeterpolynomial

$$v_k\phi_{\Gamma_1}\phi_{\Gamma_2} - Tv_{k-1}\Big(\phi_{\Gamma_1\setminus\{r\}}\phi_{\Gamma_2} + \phi_{\Gamma_1}\phi_{\Gamma_2\setminus\{s\}}\Big) + T^2 v_{k-2}\phi_{\Gamma_1\setminus\{r\}}\phi_{\Gamma_2\setminus\{s\}},$$

irrespective of the orientation of the k arrows linking Γ_1 and Γ_2. $\square$

We conclude with an example of a class of quivers which cannot be tackled with Theorem 2.2 and its corollaries:

Proposition 3.3. *If Γ is the quiver*



with $t \in \mathbb{N}$ and $k_1, \dots, k_t \in \mathbb{N}$ (the case $k_1 = \cdots = k_t = 1$ corresponding to a t-fold multiple arrow between two vertices), then

$$\phi_\Gamma = \Big(\prod_{i=1}^{t} v_{k_i}\Big)\Big((t-1)^2(T+1)^2 - t^2T - (t-2)(T+1)\sum_{i=1}^{t}\frac{v_{k_i+1}}{v_{k_i}}\Big).$$

Proof. We may assume $k_1 = \cdots = k_s = 1$ and $k_{s+1}, \dots, k_t > 1$. For $i \in \{s+1, \dots, t\}$, set

$$\Phi_i := \begin{pmatrix} & & & -1 \\ 1 & & & -1 \\ & \ddots & & \vdots \\ & & 1 & -1 \end{pmatrix} \in \mathrm{M}_{k_i-1}(\mathbb{Z}) \quad \text{and} \quad \tilde{\Phi}_i := \begin{pmatrix} & & & 0 \\ 1 & & & 0 \\ & \ddots & & \vdots \\ & & 1 & 0 \end{pmatrix} \in \mathrm{M}_{k_i-1}(\mathbb{Z}).$$

(Entries which are not shown are assumed to be zero.) Then Φ_i is the Coxetermatrix of a linear graph with $k_i - 1$ vertices and all arrows pointing in the same direction.

Counting the paths and twistpaths of Γ as in Proposition 3.1, we get

$$\Phi_\Gamma = \begin{pmatrix}
ts+t-s-1 & (t-1) & (t-1) & & (t-1) & -t \\
s+1 & \boxed{} & 1 & & 1 & -1 \\
s & \tilde{\Phi}_{s+1} & \vdots & & \vdots & \vdots \\
\vdots & & \vdots & & \vdots & \vdots \\
s & & 1 & & \vdots & \vdots \\
s+1 & 1 & \boxed{} & & \vdots & \vdots \\
s & \vdots & \tilde{\Phi}_{s+2} & & \vdots & \vdots \\
\vdots & \vdots & & & \vdots & \vdots \\
s & \vdots & & & \vdots & \vdots \\
\vdots & \vdots & 1 & \ddots & 1 & \vdots \\
s+1 & \vdots & \vdots & & \boxed{} & \vdots \\
s & \vdots & \vdots & & \tilde{\Phi}_t & \vdots \\
\vdots & \vdots & \vdots & & & \vdots \\
s & \vdots & \vdots & & & \vdots \\
s & 1 & 1 & & 1 & -1
\end{pmatrix}$$

Now consider the matrix $TE - \Phi_\Gamma$, add the last column to those corresponding to the last columns of the $\tilde{\Phi}_i$, and add the s-fold of the last column to the first. Next subtract the T-fold of the first row from the last. Finally develop the resulting determinant with respect to the last row and note that

$$\begin{vmatrix}
T-t+s+1 & 1 & 1 & & 1 \\
-1 & \boxed{TE-\Phi_{s+1}} & & & \\
-1 & & \boxed{TE-\Phi_{s+2}} & & \\
 & & & \ddots & \\
-1 & & & & \boxed{TE-\Phi_t}
\end{vmatrix}$$

is the Coxeterpolynomial of a star all arrows of which point away from the center. By Corollary 2.4, it is equal to

$$\Big(\prod_{i=s+1}^{t} v_{k_i}\Big)\Big(\sum_{i=s+1}^{t} \frac{v_{k_i+1}}{v_{k_i}} - (t-s-1)(T+1)\Big).$$

When developing the remaining determinant

$$\begin{vmatrix} 1 & 1 & & 1 & t \\ \boxed{TE-\Phi_{s+1}} & & & & 1 \\ & \boxed{TE-\Phi_{s+2}} & & & \vdots \\ & & \ddots & & \vdots \\ & & & \boxed{TE-\Phi_t} & 1 \end{vmatrix}$$

with respect to the first row, it is crucial to observe that the determinant of the matrix obtained by replacing the last column of $TE - \Phi_i$ by ${}^t(1 \ldots 1)$ is just $v_{k_i - 1}$. To simplify the resulting expression, one uses the identity

$$(T+1) - \frac{v_{k+1}}{v_k} = T\frac{v_{k-1}}{v_k}.$$

The result follows. $\square$

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