The Spectral Classes of Unicyclic Graphs

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Abstract

We study the spectral classes of a finite, connected graph without loops and with exactly one "essential" cycle. A spectral class consists of all those orientations of the graph that don't contain oriented cycles and yield the same Coxeter polynomial. We show that, if the essential cycle has m vertices, then there are exactly $[m/2]$ distinct spectral classes; the corresponding spectral radii are distinct in case the graph is wild. Furthermore, we give an explicit combinatorial expression for the entries of the powers of the Coxeter matrix of a finite quiver without oriented cycles.

1 Introduction

We consider an undirected graph Δ and we will assume throughout that Δ is connected and does not have any loops (multiple edges are allowed, though). An *essential cycle* of Δ is a full subgraph $\mathcal{C}$ of Δ with vertex set $\{x_1, \ldots, x_m\}$ where the x_i are distinct and $m \geq 3$ such that there are edges between x_i and x_{i+1} for $i = 1, \ldots, m-1$ and also between x_m and x_1. The graph Δ is called *unicyclic* in case it contains precisely one essential cycle. We are mainly interested in the collections of those orientations of the finite unicyclic graph Δ which yield the same Coxeter polynomial. These are called the *spectral classes* of Δ. It is well known and easy to show that trees admit only one spectral class; the unicyclic graphs considered here constitute the first non-trivial case.

The coefficients of the characteristic polynomial of a matrix are related to the traces of powers of that matrix. This fact induced us to search for

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an explicit formula for the entries of powers of Coxeter matrices of arbitrary quivers without oriented cycles; the result is given in section 3.

We then reduce the study of unicyclic graphs to the case where $\Delta = \mathcal{C}$ is itself an essential cycle. This case can be handled combinatorially. Returning to the general case, it turns out that a unicyclic graph whose essential cycle contains m vertices has precisely $[m/2]$ spectral classes; all of these have distinct spectral radii provided Δ is wild. Here, we call a graph *wild* if it is neither a Dynkin nor an Euclidean diagram. The only non-wild unicyclic graphs are the Euclidean diagrams $\tilde{\mathbf{A}}_{m-1}$.

In order to be able to reduce to the essential cycle case, we need the following statement, proved in section 4 using covering techniques: If Δ is unicyclic and wild, and T is a proper (but not necessarily full) subgraph without essential cycles, then for every cycle free orientation of Δ, the spectral radius of Δ is strictly bigger than that of T.

We should also mention that the coefficients of the Coxeter polynomial of a quiver are closely related to the dimensions of the Hochschild cohomology groups of the associated path algebra, see [Lu].

2 Notation and Preliminaries

2.1. We denote the set of vertices of the graph Δ by Δ_0 and its edge set by Δ_1. The *adjacency matrix* $A_\Delta = (a_{ij}) \in \mathbf{Z}^{\Delta_0 \times \Delta_0}$ of the graph Δ is the symmetric matrix whose ij-th entry is the number of edges in Δ between the vertices i and j.

For each $i \in \Delta_0$, we define a reflection

$$\sigma_i : \mathbf{R}^{\Delta_0} \longrightarrow \mathbf{R}^{\Delta_0}$$

by setting

$$e_j \sigma_i = e_j + a_{ji} e_i \quad \text{for } i \neq j \quad \text{and } e_i \sigma_i = -e_i.$$

Here $\{e_j\}_{j \in \Delta_0}$ denotes the standard basis of $\mathbf{R}^{\Delta_0}$ (i. e. $e_j(i) = \delta_{ij}$). Observe that if x and y are vertices not connected by a single edge, then $\sigma_x \sigma_y = \sigma_y \sigma_x$.

If $<$ is a total order of Δ_0, and if we write $\Delta_0 = \{y_1 < y_2 < \cdots < y_n\}$, then we call

$$\phi_{(\Delta,<)} := \sigma_{y_1} \cdots \sigma_{y_n}$$

the *Coxeter matrix* and its characteristic polynomial

$$\mathcal{X}_{(\Delta,<)}(t) = \det(tI - \phi_{(\Delta,<)})$$

the *Coxeter polynomial* belonging to Δ and $<$. We associate to Δ and $<$ the following quiver (= oriented graph) $(\Delta, <)$: The set of vertices of $(\Delta, <)$ is the set of vertices of Δ, and there are a_{ij} arrows from i to j if $i > j$ and none otherwise. Note that $(\Delta, <)$, defined in this way, has no oriented cycles, and, furthermore, every quiver without oriented cycles having Δ as its underlying graph arises in this fashion from some ordering $<$. The Coxeter matrix $\phi_{(\Delta,<)}$ and hence also the Coxeter polynomial $\mathcal{X}_{(\Delta,<)}$ depend only on the quiver $(\Delta, <)$ and not on the specific choice of $<$. The spectral radius of $\phi_{(\Delta,<)}$ will be denoted by $\rho_{(\Delta,<)}$. Recall that $\rho_{(\Delta,<)} = \max\{|\lambda| \mid \lambda \in \mathbf{C}$ is an eigenvalue of $\phi_{(\Delta,<)}\}$.

2.2. A vertex $y \in \Delta_0$ is called a *sink* of $(\Delta, <)$ if there is no arrow in $(\Delta, <)$ leaving y; similarly, y is called a *source* if there is no arrow entering y. We say that $(\Delta, <)$ has *sink-source orientation* if every vertex is either a sink or a source.

Now let $y \in \Delta_0$ be a source of $(\Delta, <)$. We denote by $r_y(\Delta, <)$ the quiver which is obtained from $(\Delta, <)$ by reversing the orientation of all the arrows containing y. In this way y becomes a sink for $r_y(\Delta, <)$.

We say that $r = r_{y_\ell} \cdots r_{y_1}$ is an *admissible change of orientation* of $(\Delta, <)$ provided that y_1 is a source of $(\Delta, <)$, $\ell \geq 1$, and y_i is a source of $r_{y_{i-1}} \cdots r_{y_1}(\Delta, <)$ for $i = 2, \ldots, \ell$.

2.3. Let $(\Delta, <)^{\mathrm{op}}$ be the quiver obtained from $(\Delta, <)$ by reversing the direction of all the arrows.

If Δ' is a subgraph of Δ (i. e. Δ' is a graph having a subset of Δ_0 as vertex set and a subset of Δ_1 as edge set), then $<$ induces a total order on Δ'_0, again denoted by $<$. The subgraph Δ' is said to be *full* if for any two vertices in Δ' the set of edges between them is the same in Δ' as in Δ. We say that Δ' is a *proper subgraph* of Δ if it is a subgraph with $\Delta' \neq \Delta$.

2.4. The following is a collection of well known results. Let Δ be a finite graph and $<$ a total order of its vertices.
(i) If $M = M_{(\Delta,<)} \in \mathbf{Z}^{\Delta_0 \times \Delta_0}$ is the matrix whose ij-th entry is equal to the number of arrows from j to i in $(\Delta, <)$, we have $A_\Delta = M + M^T$ and

$$\phi_{(\Delta,<)} = -(I - M^T)(I - M)^{-1}.$$

It follows that $\phi_{(\Delta,<)^{\mathrm{op}}} = \phi^{-1}_{(\Delta,<)}$ and also $\mathcal{X}_{(\Delta,<)} = \mathcal{X}_{(\Delta,<)^{\mathrm{op}}}$ because of $\phi_{(\Delta,<)} = (I - M)\phi^T_{(\Delta,<)^{\mathrm{op}}}(I - M)^{-1}$.

(ii) [BGP]: $\mathcal{X}_{(\Delta,<)} = \mathcal{X}_{r(\Delta,<)}$ for every admissible change of orientation r of $(\Delta, <)$.
(iii) [R]: If $x \in \Delta_0$, then there exists an admissible change of orientation r of $(\Delta, <)$ such that x is the unique source of $r(\Delta, <)$.
(iv) Assume that Δ does not contain any essential cycles and let $<'$ be another total order of Δ_0. Then there exists an admissible change of orientation of $(\Delta, <)$, say r, such that the quivers $r(\Delta, <)$ and $(\Delta, <')$ are equal. In particular, $\mathcal{X}_{(\Delta,<)} = \mathcal{X}_{(\Delta,<')}$.
(v) [Ca, PT1]: Let $(\Delta, <)$ be a quiver with n vertices, and assume that Δ does not contain essential cycles or that $<$ is a sink-source orientation. Then the Coxeter polynomial of $(\Delta, <)$ and the characteristic polynomial of the adjacency matrix of Δ are related by the following formula:

$$\mathcal{X}_{(\Delta,<)}(t^2) = t^n \det((t + t^{-1})I - A_\Delta).$$

(vi) [Bo]: Suppose there exist two full subgraphs Δ' and Δ'' of Δ such that $\Delta'_0 \cup \Delta''_0 = \Delta_0$, $\Delta'_0 \cap \Delta''_0 = \{x\}$ and $\Delta'_1 \cup \Delta''_1 = \Delta_1$. Then

$$\mathcal{X}_\Delta(t) = \mathcal{X}_{\Delta'}(t)\mathcal{X}_{\Delta''\setminus\{x\}}(t) + \mathcal{X}_{\Delta'\setminus\{x\}}(t)\mathcal{X}_{\Delta''}(t) - (t+1)\mathcal{X}_{\Delta'\setminus\{x\}}(t)\mathcal{X}_{\Delta''\setminus\{x\}}(t)$$

where $\mathcal{X}_F$ is an abbreviation for $\mathcal{X}_{(F,<)}$ and all subgraphs inherit their orientation from $(\Delta, <)$.

2.5. Let Δ be finite and let $<$ be a total order of Δ_0. It is well known that Δ is a Dynkin or Euclidean diagram if and only if $\rho_{(\Delta,<)} = 1$. We call both Δ and $(\Delta, <)$ *wild* in all other cases. The following theorem describes this situation.

Theorem. Let $(\Delta, <)$ be wild.
(i) [Ca,R]: $\rho_{(\Delta,<)}$ is a simple root of $\mathcal{X}_{(\Delta,<)}$. Moreover, $|\lambda| < \rho_{(\Delta,<)}$ for all eigenvalues $\lambda \neq \rho_{(\Delta,<)}$ of $\phi_{(\Delta,<)}$. In particular by (2.4.i), if $\mu \geq 0$ and $\mathcal{X}_{(\Delta,<)}(\mu) < 0$, then $\mu < \rho_{(\Delta,<)}$.
(ii) [PT1]: Let Δ' be a proper subgraph of Δ (not necessarily full or connected) and assume that $(\Delta, <)$ has a sink-source orientation or Δ has no essential cycle. Then $\rho_{(\Delta',<)} < \rho_{(\Delta,<)}$. $\square$

3 Iterated Coxeter Transformations

Let Δ be a finite graph and $<$ a total order of Δ_0. We are going to describe the entries of powers of the Coxeter matrix $\phi = \phi_{(\Delta,<)}$ in combinatorial terms.

Definition. A sequence $q = (p_\ell, \gamma_\ell, p_{\ell-1}, \gamma_{\ell-1}, ..., \gamma_1, p_0)$ with $\ell \geq 0$, oriented paths $p_0, ..., p_\ell$ in $(\Delta, <)$ and arrows $\gamma_1, ..., \gamma_\ell$ in $(\Delta, <)$ is called an *ℓ-twisted path from* $\mathrm{start}(p_0)$ *to* $\mathrm{end}(p_\ell)$ if $\mathrm{end}(p_{i-1}) = \mathrm{end}(\gamma_i)$ and $\mathrm{start}(p_i) = \mathrm{start}(\gamma_i)$ for $i = 1, ..., \ell$.

The sequence q is called *ℓ-endtwisted*, if in addition $0 = \mathrm{length}(p_\ell) := \#(\text{arrows belonging to } p_\ell)$ holds. If q is as above, define

$$\mathrm{length}_i(q) := \mathrm{length}(p_{i-1}) \text{ for } i = 1, ..., \ell+1 \text{ and } |q| := \ell + \sum_{i=0}^{\ell} \mathrm{length}(p_i).$$

The set of all ℓ-twisted paths in $(\Delta, <)$ from e to f is denoted by $T^\ell(e, f)$, and the subset of all ℓ-endtwisted paths from e to f is called $E^\ell(e, f)$.

3.1 Proposition. Let n be a natural number and $e, f \in \Delta_0$. Then

$$(\phi^n)_{f,e} = \sum_{\ell=1}^{n} (-1)^{n-\ell} \left(\sum_{q \in E^\ell(e,f)} \binom{|q| + n - \ell - 1}{n - \ell} - \sum_{q \in T^{\ell-1}(e,f)} \binom{|q| + n - \ell}{n - \ell} \right)$$

Proof. If we set $M = (\#\text{arrows from } j \text{ to } i)_{i,j \in \Delta_0}$ and $C = (I - M)^{-1}$, then $\phi = M^T C - C$, and an easy induction shows

$$\phi^n = (M^T C - C)^n = (\sum_{\ell=1}^{n} (-1)^{n-\ell+1} \sum_{\substack{(n_1, ..., n_\ell) \\ n_i \geq 1 \text{ with } n = \sum_{i=1}^{\ell} n_i}} (I - M^T)(C^{n_\ell}(\prod_{i=1}^{\ell-1} M^T C^{n_i}))$$

Now remember that C counts oriented paths in $(\Delta, <)$ and, more generally,

$$(C^k)_{f,e} = \sum_{p \text{ path from } e \text{ to } f} \binom{\mathrm{length}(p) + k - 1}{k - 1}.$$

Using the definition of $T^{\ell-1}(e, f)$, it follows

$$(C^{n_\ell}(\prod_{i=1}^{\ell-1} M^T C^{n_i}))_{f,e} = \sum_{q \in T^{\ell-1}(e,f)} \prod_{i=1}^{\ell} \binom{\mathrm{length}_i(q) + n_i - 1}{n_i - 1}$$

The result is now a consequence of the following identity, valid for all non negative $u, r_1, ..., r_\ell$:

$$\sum_{\substack{(u_1,\dots,u_\ell) \\ u_i \geq 0 \text{ with } u=\sum\limits_{i=1}^{\ell} u_i}} \prod_{i=1}^{\ell} \binom{r_i+u_i-1}{u_i} = \binom{(\sum\limits_{i=1}^{\ell} r_i)+u-1}{u}$$

(Choose u elements with repetition from a disjoint union of ℓ sets, the i-th of which having r_i elements). $\square$

4 Galois Coverings

4.1 Let Δ, $\bar{\Delta}$ be (not necessarily finite) graphs and let $<$ and $\bar{<}$ be total orders of their respective vertex sets. Following [DS] and [G], we say that an epimorphism of quivers $\pi : (\bar{\Delta}, \bar{<}) \longrightarrow (\Delta, <)$ is a *Galois covering defined by the group* G, if the following conditions are satisfied:
1) $G \leq \mathrm{Aut}((\bar{\Delta}, \bar{<}))$ is a group of quiver automorphisms which acts freely (i. e. the identity is the unique element of G which leaves a vertex or an arrow of $(\bar{\Delta}, \bar{<})$ fixed);
2) $\pi^{-1}(\pi x) = Gx$, for every vertex or arrow x of $(\bar{\Delta}, \bar{<})$.

Example.

$$\begin{tikzcd}[column sep=small]
\cdots & \underset{(-2,1)}{\bullet} \arrow[r] & \underset{(-1,3)}{\bullet} \arrow[r] & \underset{(-1,2)}{\bullet} & \underset{(-1,1)}{\bullet} \arrow[l] \arrow[r] & \underset{(0,3)}{\bullet} \arrow[r] & \underset{(0,2)}{\bullet} & \underset{(0,1)}{\bullet} \arrow[l] \arrow[r] & \underset{(1,3)}{\bullet} & \cdots \\
& \underset{(-2,4)}{\bullet} \arrow[u] & & & \underset{(-1,4)}{\bullet} \arrow[u] & & & \underset{(0,4)}{\bullet} \arrow[u] & &
\end{tikzcd}$$

$$\Big\downarrow \pi$$

$$\begin{tikzcd}
& & \overset{3}{\bullet} \arrow[dr] & \\
\underset{4}{\bullet} \arrow[r] & \underset{1}{\bullet} \arrow[ur] \arrow[rr] & & \underset{2}{\bullet}
\end{tikzcd}$$

where π maps each vertex (j, x) to x and each arrow $(j, x) \longrightarrow (\ell, y)$ to $x \longrightarrow y$. This is a Galois covering defined by the group $G = \{\phi_n \mid n \in \mathbf{Z}\} \simeq \mathbf{Z}$

where ϕ_n acts by $\phi_n(j,x) = (j+n, x)$.

4.2 Let $\pi : (\bar{\Delta}, \bar{<}) \longrightarrow (\Delta, <)$ be a Galois covering defined by the group G of a finite, connected quiver $(\Delta, <)$. Assume that $\bar{\Delta}$ is connected and has no essential cycles.

If $(\Delta^{(j)}, <)_{j \in \mathbf{N}}$ is a sequence of full finite subquivers of the (not necessarily finite) quiver $(\bar{\Delta}, \bar{<})$, we say that $(\Delta^{(j)}, <)_j$ has limit $(\bar{\Delta}, \bar{<})$ and write

$$(\bar{\Delta}, \bar{<}) = \lim_{j \to \infty} (\Delta^{(j)}, <)$$

if for any arrow α in $(\bar{\Delta}, \bar{<})$, there exists $N \in \mathbf{N}$ such that α is an arrow in $(\Delta^{(j)}, <)$ for all $j \geq N$.

In this situation, the limit

$$\rho_{(\bar{\Delta}, \bar{<})} := \lim_{j \to \infty} \rho_{(\Delta^{(j)}, <)}$$

exists and does not depend on the choice of the sequence $(\Delta^{(j)}, <)$. This follows from the corresponding fact about characteristic polynomials of adjacency matrices ([PT2] Theorem 1.5) and the translation mechanism provided by (2.4.v) together with (2.5.i).

Obviously, this definition does not conflict with the previously defined $\rho_{(\bar{\Delta}, \bar{<})}$ in case $\bar{\Delta}$ is itself finite. Furthermore, $\rho_{(\bar{\Delta}, \bar{<})}$ does not depend on the orientation $\bar{<}$ since the same is true for the $\rho_{(\Delta^{(j)}, <)}$ according to (2.4.iv).

4.3. Lemma. (i) Let $\pi : (\bar{\Delta}, \bar{<}) \longrightarrow (\Delta, <)$ be a Galois covering defined by the group G of a finite, connected quiver $(\Delta, <)$.

i.1) If G is finite, then $\rho_{(\bar{\Delta}, \bar{<})} = \rho_{(\Delta, <)}$.

i.2) If Δ is unicyclic and $\bar{\Delta}$ is connected and has no essential cycles, then $\rho_{(\bar{\Delta}, \bar{<})} \leq \rho_{(\Delta, <)}$.

i.3) If Δ is unicyclic and $\bar{\Delta}$ is connected and has no essential cycles, and $<$ is a sink-source orientation, then $\rho_{(\bar{\Delta}, \bar{<})} = \rho_{(\Delta, <)}$.

(ii) Let $(\Delta, <)$ be a finite quiver whose underlying graph Δ is unicyclic. Then there exists a Galois covering $(\bar{\Delta}, \bar{<}) \longrightarrow (\Delta, <)$ defined by $\mathbf{Z}$ such that $\bar{\Delta}$ is an infinite connected graph without essential cycles. Moreover, $\bar{\Delta}$ depends only on Δ and not on the orientation $<$ of Δ_0.

(iii) Let Δ be a unicyclic graph with essential cycle $\mathcal{C}$ such that $|\mathcal{C}_0|$ is even. Then Δ admits a sink-source orientation $<_0$. Moreover, whenever $<$ is another orientation of Δ, we have $\rho_{(\Delta, <_0)} \leq \rho_{(\Delta, <)}$.

Proof. Part (i.1) is [PT3] Proposition 1.5. To prove (i.2), note that $G \simeq \mathbf{Z}$ in this case, and [PT3] Proposition 1.6 applies. For the proof of (i.3): again, we have $G \simeq \mathbf{Z}$ and this is an amenable group, hence [PT2] Theorem 3.1 applies.

(ii): Let $\mathcal{C} = \{x_1, \ldots, x_m\}$ be the essential cycle of $(\Delta, <)$. We define a Galois covering $(\bar{\Delta}, \bar{<})$ of $(\Delta, <)$ as follows: $\bar{\Delta}_0 = \uplus_{\ell \in \mathbf{Z}} \Delta_0 \times \{\ell\}$, and the set $\bar{\Delta}_1((y,\ell),(z,p))$ of edges between the vertices (y,ℓ), (z,p) is given by

$$\bar{\Delta}_1((y,\ell),(z,p)) = \begin{cases} \Delta_1(y,z) & \text{if } \ell = p \text{ and } \{y,z\} \neq \{x_1, x_m\} \\ \Delta_1(x_m, x_1) & \text{if } p = \ell - 1 \text{ and } \{y,z\} = \{x_1, x_m\} \\ \emptyset & \text{otherwise} \end{cases}$$

and $\bar{<}$ is the induced orientation of $\bar{\Delta}$. This yields a Galois covering defined by $\mathbf{Z}$ similar to the one in example (4.1).

(iii): As in (ii), we can find a graph $\bar{\Delta}$ and Galois coverings $\pi : (\bar{\Delta}, \bar{<}_0) \longrightarrow (\Delta, <_0)$ and $\pi' : (\bar{\Delta}, \bar{<}) \longrightarrow (\Delta, <)$ defined by $\mathbf{Z}$. We pointed out already that $\rho_{(\bar{\Delta},\bar{<}_0)} = \rho_{(\bar{\Delta},\bar{<})}$ holds. Thus,

$$\rho_{(\Delta,<_0)} \underset{(i.3)}{=} \rho_{(\bar{\Delta},\bar{<}_0)} = \rho_{(\bar{\Delta},\bar{<})} \underset{(i.2)}{\leq} \rho_{(\Delta,<)}$$

$\square$

4.4 Proposition. Let $(\Delta', <)$ be a proper (not necessarily full) subquiver of a wild unicyclic quiver $(\Delta, <)$. Assume Δ' has no essential cycle. Then

$$\rho_{(\Delta',<)} < \rho_{(\Delta,<)}.$$

Proof. Let $\mathcal{C}$ be the essential cycle of Δ, with $m = |\mathcal{C}_0|$. By (4.3.i.1), we can assume without loss of generality that m is even. Thus, let $(\Delta, <_0)$ be a quiver of Δ with sink-source orientation. Let $\pi : (\bar{\Delta}, \bar{<}) \to (\Delta, <)$ and $\pi_0 : (\bar{\Delta}, \bar{<}_0) \to (\Delta, <_0)$ be Galois coverings defined by $\mathbf{Z}$ and $\bar{\Delta}$ connected and without essential cycles, as in (4.3.iii). Then we get

$$\rho_{(\Delta',<)} \underset{(2.4)}{=} \rho_{(\Delta',<_0)} \underset{(2.5)}{<} \rho_{(\Delta,<_0)} \underset{(4.3)}{=} \rho_{(\bar{\Delta},\bar{<}_0)} = \rho_{(\bar{\Delta},\bar{<})} \underset{(4.3)}{\leq} \rho_{(\Delta,<)}.$$

$\square$

5 The Spectral Classes of Unicyclic Graphs

In this section, Δ will be a finite, unicyclic graph with essential cycle $\mathcal{C}$. We assume throughout that $\mathcal{C}$ has m vertices $x_1, \ldots, x_m$ and that there are edges between x_i and x_{i+1} for $i = 1, \ldots, m-1$ and also between x_m and x_1.

5.1 If $<$ is a total order of Δ_0, we set

$$\begin{aligned} a := a_{(\Delta,<)} :=& \quad \#\Big\{(u,v) \in \{(x_i, x_{i+1}) \mid 1 \le i \le m-1\} \cup \{(x_m, x_1)\} \;\Big|\; u > v\Big\} \\ b := b_{(\Delta,<)} :=& \quad \#\Big\{(u,v) \in \{(x_i, x_{i+1}) \mid 1 \le i \le m-1\} \cup \{(x_m, x_1)\} \;\Big|\; v > u\Big\}, \end{aligned}$$

and define

$$v_{(\Delta,<)} := |a - b|.$$

Since $(\Delta, <)$ has no oriented cycles, both $a_{(\Delta,<)}$ and $b_{(\Delta,<)}$ are positive; furthermore, $v_{(\Delta,<)}$ does not depend on the numbering of the vertices of $\mathcal{C}$. All three numbers depend only on the quiver $(\Delta, <)$ and not on the particular total order chosen. Loosely speaking, $b_{(\Delta,<)}$ counts the number of multiarrows in $\mathcal{C}$ pointing in clockwise direction, and $a_{(\Delta,<)}$ counts the others. If Δ is equal to the Euclidean diagram $\tilde{\mathbf{A}}_{m-1}$ and $a := a_{(\tilde{\mathbf{A}}_{m-1},<)}$, $b := b_{(\tilde{\mathbf{A}}_{m-1},<)}$, we have

$$\mathcal{X}_{(\tilde{\mathbf{A}}_{m-1},<)}(t) = (t^a - 1)(t^b - 1).$$

5.2. The following theorem is the main result of the paper. The proof will follow in section (5.5).

Theorem. Let Δ be a unicyclic graph whose essential cycle $\mathcal{C}$ has m vertices.
(i) There exist integer polynomials $f, g \in \mathbf{Z}[t]$ (depending only on Δ), such that for every total order $<$ on Δ_0:

$$\mathcal{X}_{(\Delta,<)} = f + g\mathcal{X}_{(\mathcal{C},<)}.$$

Moreover, f and g are products of Coxeter polynomials of certain full subgraphs of Δ having no essential cycles.
(ii) The number of different spectral classes of Δ is equal to $[\frac{m}{2}]$ (=biggest integer less than or equal to $\frac{m}{2}$).
(iii) Let $<_1$ and $<_2$ be two total orders of Δ_0. The following statements are equivalent:

(a) $\mathcal{X}_{(\Delta,<_1)} = \mathcal{X}_{(\Delta,<_2)}$

(b) $v_{(\Delta,<_1)} = v_{(\Delta,<_2)}$

(c) there exists an admissible change of orientation r of $(\Delta,<_1)$ such that $r(\Delta,<_1) = (\Delta,<_2)$ or $r(\Delta,<_1) = (\Delta,<_2)^{\mathrm{op}}$.

Moreover, if Δ is wild, we have

$$\rho_{(\Delta,<_1)} < \rho_{(\Delta,<_2)} \Longleftrightarrow v_{(\Delta,<_1)} < v_{(\Delta,<_2)}.$$

Part (ii) of this theorem was proved by Coleman in [C] in case $\Delta = \mathcal{C}$ is itself an essential cycle.

5.3. Lemma: (i) $a_{(\Delta,<)^{\mathrm{op}}} = m - a_{(\Delta,<)}$ and therefore $v_{(\Delta,<)} = v_{(\Delta,<)^{\mathrm{op}}}$.
(ii) Let $(\Delta,<_1)$ and $(\Delta,<_2)$ be quivers of Δ. We have $v_{(\Delta,<_1)} = v_{(\Delta,<_2)}$ if and only if there exists an admissible change of orientation r of $(\Delta,<_1)$ such that $r(\Delta,<_1) = (\Delta,<_2)$ or $r(\Delta,<_1) = (\Delta,<_2)^{\mathrm{op}}$. In this case, we have $\mathcal{X}_{(\Delta,<_1)} = \mathcal{X}_{(\Delta,<_2)}$.

Proof. (i) is clear.

(ii) "$\Leftarrow$" By (i) and induction, it is enough to take $r = r_x$, where $x \in \Delta_0$ is a source of $(\Delta,<_1)$, and show that $v_{r(\Delta,<_1)} = v_{(\Delta,<_1)}$. This is clear if $x \notin \mathcal{C}_0$ because the edges in $\mathcal{C}$ are not affected by the application of r. If, on the other hand, x is a vertex of $\mathcal{C}$, then the orientation of those edges of $\mathcal{C}$ that contain x will change, but the numbers $a_{(\Delta,<_1)}$, $b_{(\Delta,<_1)}$ and hence $v_{(\Delta,<_1)}$ remain the same.

"$\Rightarrow$" In the first case, we consider the situation when $a_{(\Delta,<_1)} = a_{(\Delta,<_2)}$. Pick $x \in \mathcal{C}_0$ arbitrary. We can find admissible changes of orientation s,t of $(\Delta,<_1)$ and $(\Delta,<_2)$ so that x is the unique source of both $s(\Delta,<_1)$ and $t(\Delta,<_2)$, and therefore of $\mathcal{C}$, according to (2.4.iii). We have $a_{s(\Delta,<_1)} = a_{t(\Delta,<_2)}$ and it is then clear that $s(\Delta,<_1)$ and $t(\Delta,<_2)$ must be the same quivers, which provides us with an admissible change of orientation r of $(\Delta,<_1)$ such that $r(\Delta,<_1) = (\Delta,<_2)$.

In the case $a_{(\Delta,<_1)} = b_{(\Delta,<_2)} = a_{(\Delta,<_2)^{\mathrm{op}}}$, using the same arguments, we can exhibit an admissible change of orientation r of $(\Delta,<_1)$ such that $r(\Delta,<_1) = (\Delta,<_2)^{\mathrm{op}}$. $\square$

5.4 Proposition. For $i = 1, ..., [\frac{m}{2}]$, let $(\mathcal{C},<_i)$ be the orientation of $\mathcal{C}$ with unique source x_1 and $a_{(\mathcal{C},<_i)} = i$. We write $a_{ij} := a_{x_i x_j}$ for the number of edges between the vertices x_i and x_j and set $d := a_{12}a_{23}\ldots a_{(m-1)m}a_{m1}$. Set

$\mathcal{X}_i := \mathcal{X}_{(\mathcal{C},<_i)}$. Then

$$\mathcal{X}_i(t) - \mathcal{X}_j(t) = d(t^j + t^{m-j} - t^i - t^{m-i}),$$

and, in particular, the $\mathcal{X}_i$ are pairwise distinct for $1 \leq i \leq [\frac{m}{2}]$.
Proof: Write $M_i := M_{(\Delta,<_i)}$. We then have

$$\mathcal{X}_i(t) = \det(t(I - M_i) + I - M_i^T),$$

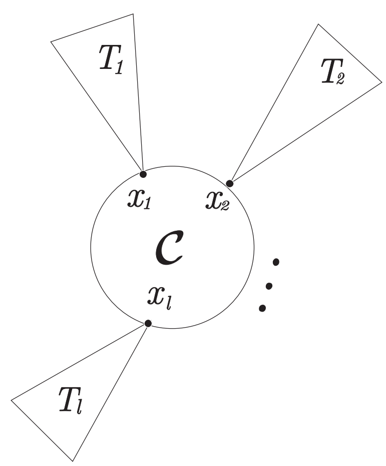
since $\det(I - M_i) = 1$. Observe that $(t(I-M_i)+I-M_i^T)_{uv} = t+1$ if $u = v$, and $-[t(\#\text{arrows}(v \to u)) + (\#\text{arrows}(u \to v))]$ if $u \neq v$.

Then, by the Leibniz formula for the determinant, it follows

$$\mathcal{X}_i(t) = \left(\sum_{\sigma \in S_m \setminus \{(1,2,\ldots,m),(m,m-1,\ldots,1)\}} \operatorname{sgn}(\sigma)(t+1)^{m-2\ell_\sigma} t^{\ell_\sigma} a^2_{i_1^{(\sigma)} j_1^{(\sigma)}} \cdots a^2_{i_{\ell_\sigma}^{(\sigma)} j_{\ell_\sigma}^{(\sigma)}} \right) \\ -d(t^i + t^{m-i})$$

where $\sigma = (i_1^{(\sigma)} j_1^{(\sigma)}) \cdots (i_{\ell_\sigma}^{(\sigma)} j_{\ell_\sigma}^{(\sigma)})$ is a minimal expression of σ as a product of transpositions and S_m denotes the group of permutations of the set $\{1, \ldots, m\}$. $\square$

5.5. Proof of Theorem (5.2): (i) Since Δ is a unicyclic graph, it has the following shape:



where all T_i are trees and $x_i \in \mathcal{C}_0 \cap (T_i)_0$, $i = 1, ..., \ell$. Then the result follows by induction on ℓ and (2.4.vi), taking into account that the Coxeter polynomial of a graph without essential cycles does not depend on its orientation.

(ii) follows from (iii), below.
(iii) The equivalence of (b) and (c) was proved in Lemma (5.3.ii), while (c)$\Rightarrow$(a) follows from (2.4). To see (a)$\Rightarrow$(b), assume $v_{(\Delta,<_1)} \neq v_{(\Delta,<_2)}$. Using (2.4.iii), we find admissible changes of orientation r and s so that both $r(\mathcal{C},<_1)$ and $s(\mathcal{C},<_2)$ have unique source x_0. Because of $v_{r(\mathcal{C},<_1)} \neq v_{s(\mathcal{C},<_2)}$, Proposition (5.4) together with (i) shows that $\mathcal{X}_{(\Delta,<_1)} \neq \mathcal{X}_{(\Delta,<_2)}$.

Now assume Δ is wild, and $v_{(\Delta,<_1)} < v_{(\Delta,<_2)}$. Without loss of generality, we can assume that $a_{(\Delta,<_2)} \leq b_{(\Delta,<_2)}$. Since $v_{(\Delta,<_1)} < v_{(\Delta,<_2)}$, there exist numbers $\alpha, \beta > 0$ such that $a_{(\Delta,<_1)} = a_{(\Delta,<_2)} + \alpha$, $b_{(\Delta,<_1)} = a_{(\Delta,<_2)} + \beta$, thus $b_{(\Delta,<_2)} = a_{(\Delta,<_2)} + \alpha + \beta$. Write $\rho := \rho_{(\Delta,<_1)}$. We then get

$$\begin{aligned} \mathcal{X}_{(\mathcal{C},<_2)}(\rho) - \mathcal{X}_{(\mathcal{C},<_1)}(\rho) &\underset{(5.4)}{=} d(\rho^{a(\Delta,<_1)} + \rho^{b(\Delta,<_1)} - \rho^{a(\Delta,<_2)} - \rho^{b(\Delta,<_2)}) \\ &= -d\rho^{a(\Delta,<_2)}(\rho^{\alpha} - 1)(\rho^{\beta} - 1) < 0 \end{aligned}$$

since $\rho > 1$ (2.5). Note that $g(\rho) > 0$, where g is the polynomial from (i); this follows from (4.4) and (2.5.i). We get:

$$\mathcal{X}_{(\Delta,<_2)}(\rho) = \mathcal{X}_{(\Delta,<_2)}(\rho) - \mathcal{X}_{(\Delta,<_1)}(\rho) = g(\rho)(\mathcal{X}_{(\mathcal{C},<_2)}(\rho) - \mathcal{X}_{(\mathcal{C},<_1)}(\rho)) < 0$$

which, by (2.5.i), implies $\rho < \rho_{(\Delta,<_2)}$. □

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