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Two Aspects of Finite-Dimensional Algebras: Uniserial Modules and Coxeter Polynomials

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ABSTRACT

Two Aspects of Finite-Dimensional Algebras:
Uniserial Modules and Coxeter Polynomials

by

Axel Boldt

In this dissertation, we define "triangles of uniserials" and locate them in the Auslander-Reiten quivers of finite-dimensional triangular algebras. Irreducible morphisms between uniserial modules over hereditary algebras are also classified, as well as those uniserial modules having uniserial Auslander-Reiten translate. It is shown that, over a wild hereditary algebra, almost all uniserial modules are regular and quasi-simple.

Moreover, we give a reduction principle for calculating Coxeter polynomials and use it to determine the spectral classes of unicyclic graphs. Combining this principle with covering techniques, we show that, if the cycle of a unicyclic graph consists of m points, then the graph has $[m/2]$ spectral classes and, in case the graph is wild, the spectral radii of these classes are distinct.

Contents

Introduction

This dissertation contains the study of two aspects of the representation theory of finite-dimensional algebras: Uniserial modules over triangular and hereditary algebras and the patterns in which they arise as vertices of the Auslander-Reiten quiver in part 1, and the Coxeter polynomials of hereditary algebras and, in particular, algebras based on unicyclic quivers in part 2.

By Λ we will denote a finite-dimensional associative algebra over a field K, by J its Jacobson radical, and the modules we consider will be finitely generated left Λ-modules.

In the first part, we focus on *uniserial modules*, i. e. those non-zero modules that admit precisely one composition series. These are (arguably) the simplest indecomposables, and this makes it interesting to understand their rôle within the category Λ- mod of finitely generated left Λ-modules. One major tool used to study this latter category is its *Auslander-Reiten quiver*, the directed graph having as vertices the isomorphism classes of indecomposable modules and as arrows the irreducible maps between them; a homomorphism $f : M \longrightarrow N$ between indecomposable modules M and N is called *irreducible* if the only possible factorizations $f = gh$ are trivial, i. e. either h is a split monomorphism or g is a split epimorphism. These irreducible maps are important because, in many situations, they constitute the basic building blocks for arbitrary homomorphisms.

Another motivation for studying irreducible homomorphisms lies in their tight connection to Auslander-Reiten sequences. A non-split short exact sequence

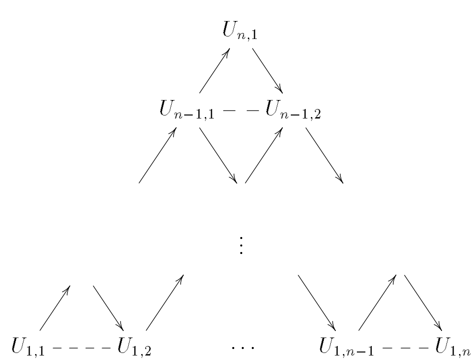
$$0 \longrightarrow M \xrightarrow{f} X \xrightarrow{g} N \longrightarrow 0$$

with indecomposable end terms M and N is called an *Auslander-Reiten sequence* if every homomorphism $h : L \longrightarrow N$ that is not a split epimorphism can be lifted over g. It turns out that, up to isomorphism, there is always precisely one Auslander-Reiten sequence ending in a non-projective indecomposable module N. Its initial term is called the *Auslander-Reiten translate* of

N and denoted by $\tau N = M$. The situation is symmetric in that all indecomposable non-injective modules arise uniquely in the form τN. Because τ can be computed rather easily, this provides a method for constructing new indecomposable modules from known ones. The connection with irreducible maps is as follows: if the middle term X is decomposed into a direct sum of indecomposables, then the corresponding components of f represent all the irreducible maps starting in M and the components of g represent all the irreducible maps ending in N.

In Chapter 1, we focus on split basic *triangular* algebras. These are the homomorphic images of finite-dimensional path algebras $K\Gamma$, where Γ is a finite quiver without oriented cycles (see e. g. [**17**, sec. 2.1] for the definitions, but note that, in contrast to the convention adopted there, we compose paths like maps: if p is a path from x to y and q is a path from y to z, then we denote the composite path from x to z by qp). Our primary goal is to understand irreducible maps $f : U_1 \longrightarrow U_2$ between uniserial modules U_1, U_2. The following basic fact can easily be proved for arbitrary algebras: namely, the only irreducible maps between uniserial modules are certain radical embeddings $JU \longrightarrow U$ and socle factor projections $U \longrightarrow U/\operatorname{soc} U$; since these are clearly dual to each other, our question therefore becomes: For which uniserial modules U over a triangular algebra Λ is the radical embedding $JU \longrightarrow U$ irreducible? We provide a sufficient as well as a necessary combinatorial condition in terms of quivers and relations for this phenomenon. The two conditions are separated by a rather slim margin; but an elimination of this gap has unfortunately not yet been accomplished. However, the existing results suffice to completely characterize the irreducible maps between uniserials over hereditary algebras, and to locate and identify certain interesting patterns of uniserials in the Auslander-Reiten quiver of general triangular algebras. We call these "triangles of uniserials".

A *triangle of uniserials* is defined to be a full subquiver of the Auslander-Reiten quiver of the following shape:



where $n \in \mathbb{N}$ and the U_{ij} are pairwise non-isomorphic uniserial left Λ-modules with

$$\operatorname{length}(U_{ij}) = i \quad \text{for all } i \text{ and } j.$$

The notation $U - - - V$ indicates that $\tau V = U$. It turns out that these triangles occur quite frequently. For example if $\vec{\mathbb{A}}_n$ is the linearly oriented graph $\mathbb{A}_n$,

$$\bullet \longrightarrow \bullet \longrightarrow \cdots \longrightarrow \bullet\,,$$

then all irreducible $K\vec{\mathbb{A}}_n$-modules are uniserial, and the full Auslander-Reiten quiver of $K\vec{\mathbb{A}}_n$ is a triangle of uniserials. So, whenever a triangle of uniserials occurs in the Auslander-Reiten quiver of Λ, one can interpret this as occurrence of a subcategory $\mathcal{T}$ equivalent to a category of the form $K\vec{\mathbb{A}}_n$-mod inside Λ-mod; in particular, each such subcategory $\mathcal{T}$ has Auslander-Reiten sequences.

The main theorem of Chapter 1 completely characterizes those uniserial modules over triangular algebras that appear as the upper tips of maximal triangles of uniserials. The proof relies on the information about irreducible maps between uniserial modules that was obtained earlier.

In Chapter 2, then, we specialize to the hereditary case, i. e., we study modules over finite-dimensional path algebras $K\Gamma$. In fact, it was the hereditary case that provided the original motivation for the work on triangular algebras in the first place. As we already mentioned earlier, in this situation it is possible to characterize the irreducible maps between uniserial modules completely. Moreover, the resulting description of triangles is much more transparent than in the triangular case. In particular, we manage to give a description of the tips of triangles that does not involve any combinatorics but is completely module-theoretic. As a consequence, virtually all irreducible maps between uniserials (more precisely: all but the radical embeddings of projectives and the socle factor projections of injectives) are embeddable in triangles of uniserials. This means that, except for the projectives and the injectives, the uniserial modules appear in the Auslander-Reiten quiver either isolated or in triangles. Combined with the standard information about the structure of Auslander-Reiten quivers of hereditary algebras, this yields still stronger insight about the possible location of uniserial modules inside that quiver: we show that, in the wild case, almost all uniserial modules are regular and quasi-simple. An indecomposable module X is called *regular* if its connected component in the Auslander-Reiten quiver does not contain any projective or injective modules; it is called *quasi-simple* if there does not exist any injective irreducible map ending in X.

One of the reasons for this clean picture lies in the fact that the hereditary situation allows for an easy description of the uniserials: Modules over the algebra $K\Gamma$ are simply K-linear representations of the quiver Γ, i. e. assignments of finite-dimensional K-vectorspaces to the vertices and K-linear maps to the arrows of Γ. As a consequence, one obtains all uniserial modules by choosing a path in Γ (sometimes called a "mast" in this situation), by assigning the vectorspace K to every vertex on the mast and 0 to all others, and by substituting the identity map for each of the arrows on the mast. Using this explicit description, we classify (in terms of their masts) those uniserials modules U over $K\Gamma$ which have uniserial Auslander-Reiten translate τU; in the positive case, this uniserial module τU is pinned down in terms of its representation.

In general, if we have a short exact sequence

$$0 \longrightarrow U \longrightarrow X \longrightarrow V \longrightarrow 0$$

with uniserial end terms U and V, then the middle term X is either indecomposable or a direct sum of two uniserials. Combining the information about irreducible maps and Auslander-Reiten translates, one can easily describe those Auslander-Reiten sequences over hereditary algebras which are made up from four uniserial modules. Another application is as follows: it happens comparatively often that every uniserial U has the property that either the sequence $\tau^i U$ for $i \geq 0$ consists only of uniserials and ends eventually in a projective module, or the sequence $\tau^{-i} U$ for $i \geq 0$ consists only of uniserials and ends in an injective module. In other words, all uniserials are connected to projectives or injectives via a chain of τ-translations that only involve uniserials. We classify those hereditary algebras of finite representation type for this is always true.

In the second part, we examine the Coxeter polynomial of a hereditary algebra $\Lambda = K\Gamma$. The *Coxeter polynomial* is the characteristic polynomial of the Coxeter transformation, which in turn is the unique endomorphism Φ of the Grothendieck group $K_0(\Lambda)$ satisfying

$$\Phi[P_i] = -[I_i] \quad \text{for all vertices } i \text{ of } \Gamma.$$

Here, $[X]$ denotes the image of the module $X \in \Lambda\text{-mod}$ in $K_0(\Lambda)$ (which can be identified with the dimension vector of the associated representation of Γ) and P_i resp. I_i are the indecomposable projective resp. injective modules corresponding to the vertex i.

The significance of the Coxeter transformation stems from the fact that it describes the Auslander-Reiten translation on the level of the Grothendieck group:

$$\Phi[X] = [\tau X] \quad \text{for every non-projective indecomposable } X.$$

The Coxeter polynomial – especially its zero set – thus encodes important information about the asymptotic growth behavior of the Auslander-Reiten translation. But it also contains homological information: its coefficients are closely related to the dimensions of the Hochschild cohomology groups of Λ, see [**16**].

The coefficients of the characteristic polynomial of a matrix, in general, can easily be calculated from the traces of powers of that matrix [**13**, p.87]. This induced us to search for an explicit formula for the entries of powers of the Coxeter matrix in terms of the combinatorics of the quiver Γ. The result can be found in Chapter 4.

In addition, we present a reduction formula that significantly simplifies the calculation of Coxeter polynomials in concrete situations: in the situation where the quiver Γ is the union of two subquivers Γ_1 and Γ_2 that have precisely one vertex in common, we can express the Coxeter polynomial of Γ in terms of the Coxeter polynomials of these smaller quivers. This trick quite naturally leads to explicit formulas for the Coxeter polynomials of certain classes of quivers; these were already given in [**4**], but we include them here in order to illustrate the method. In fact, we were able to prove a more general version of this reduction principle which even allows for the presence of certain types of relations on the quiver Γ. This work has been published in [**5**].

In the sequel, we approach the following problem: given an undirected graph Δ, how many orientations of Δ yield different Coxeter polynomials? In this context, the collection of all the orientations that result in the same Coxeter polynomial is called a *spectral class of* Δ. It is well known that trees have only a single spectral class. We solve the first non-trivial case: namely, we deal with those graphs Δ which contain exactly one cycle; these graphs we call *unicyclic.*

The statement of the main theorem is as follows: If the cycle of a unicyclic graph Δ consists of m points, then Δ has $[m/2]$ spectral classes. Moreover, if the graph Δ is wild (which, in this situation, means that it does not only consist of a cycle by itself), then the different spectral classes have different spectral radii (defined as the spectral radii of the corresponding Coxeter polynomials). We completely describe the spectral classes and order them according to their spectral radii. The proof of this theorem uses the reduction formula established earlier as well as covering techniques.

This last part of the thesis, addressing spectral classes of unicyclic graphs, is the result of joint work with Martha Takane, see [**6**].

Part 1

Patterns of Uniserial Modules in the Auslander-Reiten Quiver of Finite-Dimensional Algebras

CHAPTER 1

Uniserial Modules over Triangular Algebras

1.1. Introduction and General Results

In this chapter, we will study uniserial modules over triangular algebras, with emphasis on the irreducible maps between them and "triangles of uniserials" (certain full translation subquivers of the Auslander-Reiten quiver isomorphic to the Auslander-Reiten quivers of linearly oriented graphs $\mathbb{A}_n$).

Section 2 first gives a necessary combinatorial condition for the radical embedding of a uniserial module over a triangular algebra to be irreducible, followed by a sufficient condition. Unfortunately, a complete combinatorial characterization of this phenomenon has not yet been accomplished; however, the present results suffice to completely understand triangles of uniserials. The theorems concerning triangles are presented in Section 3. In addition, Section 3 gives an example of a phenomenon that cannot appear in the hereditary case: "incomplete triangles". These examples can easily be constructed by means of the results of Section 2.

In the beginning of this introductory section, we will present some rather elementary general results about uniserial modules and then set up the notation to be used in the analysis of the triangular case. We will also emphasize the differences between the general triangular and the hereditary theories.

Definition 1.1.1. Let R be a ring. A non-zero R-module U is called *uniserial* if the lattice of its submodules forms a chain, i. e. if every two submodules of U are comparable.

Obviously, all subfactors of a uniserial module are again uniserial.

In the sequel, we will exclusively deal with uniserial left modules U of finite length. The lattice of proper non-zero submodules of U is then a finite chain with maximal element $\operatorname{rad} U$ and minimal element $\operatorname{soc} U$. If, moreover, the ring R is left artinian with Jacobson radical J, then the submodules of U are given by $J^l U$, $l = 0, \dots, \operatorname{length}(U)$.

Remark. Let R be left artinian with Jacobson radical J and let $X \in R\text{-}\operatorname{mod}$. If X/JX is simple, then every $x \in X \setminus JX$ generates X.

PROOF. We have $X = Rx + JX$ by hypothesis. Since JX is small in X, we obtain $X = Rx$. □

PROPOSITION 1.1.2. *Let R be a left artinian ring and consider a short exact sequence*

$$0 \longrightarrow U_1 \xrightarrow{f} M \xrightarrow{g} U_2 \longrightarrow 0$$

in R-mod with uniserial modules U_1 and U_2. Then M is either indecomposable or a direct sum of two uniserial modules.

PROOF. We will again denote the Jacobson radical of R by J. Assume we have a decomposition $M = M_1 \oplus M_2$ with both M_1 and M_2 non-zero. Decompose f and g accordingly, i. e., write $f = \binom{f_1}{f_2}$ and $g = (g_1, g_2)$, and let

$$\bar{\ }: R\text{-mod} \longrightarrow (R/J)\text{-mod}$$

be the functor $R/J \underset{R}{\otimes} -$. We then get the right exact sequence

$$\bar{U_1} \xrightarrow{\binom{\bar{f_1}}{\bar{f_2}}} \bar{M_1} \oplus \bar{M_2} \xrightarrow{(\bar{g_1}, \bar{g_2})} \bar{U_2} \longrightarrow 0$$

where $\bar{U_1}$ and $\bar{U_2}$ are simple and $\bar{M_1}$, $\bar{M_2}$ non-zero semisimple. Comparing the lengths of the involved modules, we see that both $\bar{M_1}$ and $\bar{M_2}$ must be simple and $\bar{f} \neq 0$. Without loss of generality, we may assume $\bar{f_1}(\bar{U_1}) = \bar{M_1}$.

Pick $u_1 \in U_1 \setminus JU_1$. Then $f_1(u_1) \in M_1 \setminus JM_1$ generates M_1. Hence f_1 is surjective and M_1 is uniserial. If $f_2(u_1) = 0$, then $f_2 = 0$ and g_2 is injective, and consequently M_2 is uniserial. If $f_2(u_1) \neq 0$, we can find $l \geq 0$ with $f_2(u_1) \in J^l M_2 \setminus J^{l+1} M_2$. If $l = 0$, then $f_2(u_1)$ generates M_2 and M_2 is therefore uniserial. We will assume $l > 0$ from now on.

Claim 1: $\mathrm{im}(g_1) \subset J^l U_2$.

Let $m_1 \in M_1$; write $m_1 = \alpha f_1(u_1) = f_1(\alpha u_1)$ with $\alpha \in \Lambda$. Then $g_1(m_1) = g(m_1) = g f_1(\alpha u_1) - g f(\alpha u_1) = -g f_2(\alpha u_1) \subset g(J^l M_2) \subset J^l U_2$. Hence we have $g_1(M_1) \subset J^l U_2$.

Claim 2: *g_2 is surjective and the map $M_2/J^l M_2 \longrightarrow U_2/J^l U_2$ induced by g_2 is an isomorphism.*

Let $m_2 \in M_2 \setminus JM_2$. Then $u_2 := g_2(m_2) \in U_2 \setminus JU_2$ (since $g_2(m_2) \in JU_2$ would imply $\mathrm{im}(g) = \mathrm{im}(g_1) + \mathrm{im}(g_2) \subset J^l U_2 + JU_2 \subsetneq U_2$, a contradiction). Since u_2 generates U_2, g_2 is surjective. Now let $x \in M_2 \setminus J^l M_2$ and assume $g_2(x) \in J^l U_2$, say $g_2(x) = \alpha u_2 = g_2(\alpha m_2)$ with $\alpha \in J^l$. Then $x - \alpha m_2 \in \mathrm{kern}(g_2) \setminus J^l M_2 \subset \mathrm{im}(f_2) \setminus J^l M_2 = \varnothing$, again a contradiction.

Claim 3: *$J^l M_2$ is uniserial.*

By restricting our maps f and g, we obtain the following short exact sequence:

$$0 \longrightarrow U_1 \longrightarrow M_1 \oplus J^l M_2 \longrightarrow J^l U_2 \longrightarrow 0$$

and we see as above that $J^l M_2 / J^{l+1} M_2$ is simple, hence $J^l M_2$ is generated by $f_2(u_1)$ and $f_2 : U_1 \longrightarrow J^l M_2$ is therefore surjective.
Claim 4: *M_2 is uniserial.*
We know that $J^k M_2 / J^{k+1} M_2$ is simple or 0 for all $k \in \mathbb{N}$. □

PROPOSITION 1.1.3. *Let R be a left artinian ring with Jacobson radical J. If $U_1, U_2 \in R\text{-}\mathrm{mod}$ are uniserial and $f : U_1 \longrightarrow U_2$ is an irreducible R-linear map, then either*
(1) *there exists an isomorphism $\phi : JU_2 \longrightarrow U_1$ so that $f\phi$ is the natural radical embedding $JU_2 \longrightarrow U_2$, or*
(2) *there exists an isomorphism $\psi : U_2 \longrightarrow U_1/\operatorname{soc} U_1$ so that ψf is the natural socle factor projection $U_1 \longrightarrow U_1/\operatorname{soc} U_1$.*

PROOF. We only consider the case where f is injective; the case of a surjection is analogous. We know that $\mathrm{im}(f)$ is a proper submodule of U_2, hence $\mathrm{im}(f) = J^l U_2$ with $l \geq 1$ and $U_1 \simeq J^l U_2$ via f. However, if $l > 1$, then $J^l U_2 \longrightarrow J^{l-1} U_2 \longrightarrow U_2$ would clearly be a non-trivial factorization of $J^l U_2 \longrightarrow U_2$ and would yield such a factorization of f, which is impossible. □

We will now leave the general situation and concentrate on the case of a triangular algebra. Let K be a field and let $\Lambda = K\Gamma/I$ be a finite dimensional, triangular K-algebra (i. e. Γ is a finite quiver *without oriented cycles* and I is an admissible ideal of relations in the path algebra $K\Gamma$). The Jacobson radical of Λ will be denoted by J.

We will write $\mathrm{A}\Gamma$ for the set of arrows of Γ and $\mathrm{V}\Gamma$ for the set of vertices of Γ. Arrows in Γ will be identified with their images in Λ; moreover, we will write e_x for the primitive idempotent in Λ corresponding to the vertex $x \in \mathrm{V}\Gamma$. The starting vertex of an arrow $\alpha \in \mathrm{A}\Gamma$ will be denoted by $\mathrm{s}(\alpha)$ and its terminal vertex by $\mathrm{t}(\alpha)$. The same notation will be used for starting and terminating vertices of paths. We will furthermore identify left Λ-modules with the corresponding representations of Γ. Note however that not all representations of Γ arise in this fashion, due to the relations in I. Whenever $V = ((V_x)_{x \in \mathrm{V}\Gamma}, (g_\alpha)_{\alpha \in \mathrm{A}\Gamma})$ is a representation of Γ and $\mu \in e_x K\Gamma e_y$ with $x, y \in \mathrm{V}\Gamma$, we will write g_μ for the induced linear map $V_y \longrightarrow V_x$. Finally, K-linear maps having domain K will be identified with their values at 1.

A major tool for the combinatorial treatment of uniserials is given in the following definition:

Definition 1.1.4. A path p in Γ is called a *mast* of the uniserial module $U \in \Lambda\text{-}\operatorname{mod}$ if $\operatorname{length}(p) = \operatorname{length}(U) - 1$ and $pU \neq 0$.

Remark. Every uniserial $U \in \Lambda\text{-}\operatorname{mod}$ has a mast.

Proof. If $n := \operatorname{length}(U)$, then $0 \neq \operatorname{soc} U = J^{n-1}U$ and J^{n-1} is generated by the images of the paths of length $n-1$. □

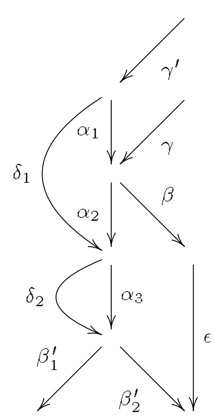
To prepare for our analysis in the following sections, we fix a finitely generated uniserial left Λ-module U with mast

$$p = 1 \xrightarrow{\alpha_1} 2 \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_{n-1}} n.$$

On several occasions, we will refer to certain subpaths $\alpha_i \cdots \alpha_j$ of p; whenever $i < j$, this expression will simply stand for 1. We now name all the arrows in Γ that touch p, classifying them according to the type of contact with p.

$$\begin{aligned}
B \ &:= \Big\{\beta \ \in A\Gamma \mid \mathrm{s}(\beta) \in \{1, \ldots, n-1\} \text{ and } \mathrm{t}(\beta) \notin \{1, \ldots, n\}\Big\},\\
B' &:= \Big\{\beta' \in A\Gamma \mid \mathrm{s}(\beta') = n\Big\},\\
C \ &:= \Big\{\gamma \ \in A\Gamma \mid \mathrm{s}(\gamma) \notin \{1, \ldots, n\} \text{ and } \mathrm{t}(\gamma) \in \{2, \ldots, n\}\Big\},\\
C' &:= \Big\{\gamma' \in A\Gamma \mid \mathrm{t}(\gamma') = 1\Big\},\\
D \ &:= \Big\{\delta \ \in A\Gamma \mid \{\mathrm{s}(\delta), \mathrm{t}(\delta)\} \subset \{1, \ldots, n\} \text{ and } \delta \notin \{\alpha_1, \ldots, \alpha_{n-1}\}\Big\}.
\end{aligned}$$

For an illustration of these definitions with an example, consider the following quiver Γ, together with the path $p = \alpha_3\alpha_2\alpha_1$:



We then have

$$\begin{aligned} B &= \{\beta\}, \\ B' &= \{\beta_1', \beta_2'\}, \\ C &= \{\gamma\}, \\ C' &= \{\gamma'\}, \\ D &= \{\delta_1, \delta_2\}. \end{aligned}$$

Observe that, in general, our uniserial module U may be identified with a representation $U = ((U_x), (f_\alpha))$ of Γ, where

$$U_x = \begin{cases} K, & \text{if } x \in \{1, \dots, n\}; \\ 0, & \text{else} \end{cases}$$

and

$$f_{\alpha_i} = \mathrm{id} \quad \text{for every } i \in \{1, \dots, n-1\}.$$

The module U is then completely determined by the choice of the mast p and the scalars $f_\delta(1)$ for $\delta \in D$, different sets of scalars corresponding to non-isomorphic modules.

Unlike the hereditary case, not every path is a mast, however, and not every set of scalars appears in this fashion, since the relations in I impose restrictions. For instance if, in the example above, the relations

$$\delta_2\delta_1 + \alpha_3\alpha_2\alpha_1 + \alpha_3\delta_1 = 0 = \epsilon\beta - \beta_2'\alpha_3\alpha_2,$$

are imposed, then the path $\epsilon\beta$ is not the mast of any uniserial module. Moreover, it is easy to see that the allowable sets of scalars (once a mast is fixed) are the points of a certain affine variety over K; in our example, the variety for the mast $p = \alpha_3\alpha_2\alpha_1$ is $\{(x, y) \in K^2 \mid xy + 1 + y = 0\}$.

1.2. Irreducible Radical Embeddings of Uniserials

We know from 1.1.3 that, in order to understand irreducible maps between uniserial modules, it is sufficient to study radical embeddings (and their duals, socle factor projections). The following conjecture covers this situation; we manage to prove "(2) $\Rightarrow$ (1)" and "(1) $\Rightarrow$ (2)(a)" in the sequel. These two implications, together with their duals, will suffice to completely characterize triangles of uniserials in the next section.

CONJECTURE 1.2.1. *The following statements are equivalent:*

(1) *The embedding $JU \longrightarrow U$ is irreducible.*

(2) *U is not simple and satisfies both* (a) *and* (b) *below:*

(a) *For every $\beta \in B$,*

$$\beta\alpha_{s(\beta)-1} \cdots \alpha_1 \in Jp,$$

and for every $\delta \in D$,

$$\delta\alpha_{\mathrm{s}(\delta)-1}\cdots\alpha_1 \in K\alpha_{\mathrm{t}(\delta)-1}\cdots\alpha_1.$$

(b) *There exists a subset* $R \subset J$ *such that* $\{rp + J^2p \mid r \in R\}$ *forms a* K*-basis for* Jp/J^2p *and* (i) *and* (ii) *both hold:*

(i) *For every* $\gamma \in C$ *there exists* $w \in pJ$ *such that, for every* $r \in R$,

$$r\alpha_{n-1}\cdots\alpha_{\mathrm{t}(\gamma)}\gamma = rw.$$

(ii) *For every* $\delta \in D$ *and every* $r \in R$,

$$r\alpha_{n-1}\cdots\alpha_{\mathrm{t}(\delta)}\delta \in Kr\alpha_{n-1}\cdots\alpha_{\mathrm{s}(\delta)}.$$

Proof of "(2) $\Rightarrow$ (1)". Let $V = ((V_x),(g_\alpha)) \in \Lambda\text{-}\mathrm{mod}$ and suppose there exist Λ-linear maps

$$JU \xrightarrow{\Phi=(\Phi_x)} V \xrightarrow{\Psi=(\Psi_x)} U$$

such that $\Psi\Phi$ is equal to the embedding $JU \longrightarrow U$.
Observe that we can assume without loss of generality that the elements of the set R arising from condition (2) are normed in the following fashion: $r = e_{u(r)}re_n$ for certain vertices $u(r) \in \mathrm{V}\Gamma$. We can thus denote by g_r the K-linear map $V_n \longrightarrow V_{u(r)}$ induced by left multiplication with r.

Note furthermore that we can strengthen the conditions on $\delta \in D$ in the following manner:

$$\delta\alpha_{\mathrm{s}(\delta)-1}\cdots\alpha_1 = f_\delta(1)\alpha_{\mathrm{t}(\delta)-1}\cdots\alpha_1$$

and for every $r \in R$

$$r\alpha_{n-1}\cdots\alpha_{\mathrm{t}(\delta)}\delta = f_\delta(1)r\alpha_{n-1}\cdots\alpha_{\mathrm{s}(\delta)}.$$

The first equation is clear, and the second one follows then from

$$r\alpha_{n-1}\cdots\alpha_{\mathrm{t}(\delta)}\delta\alpha_{\mathrm{s}(\delta)-1}\cdots\alpha_1 = f_\delta(1)r\alpha_{n-1}\cdots\alpha_1$$

since $rp \neq 0$ for $r \in R$.
Case 1: *There exists* $v \in V_1$ *with* $\Psi_1(v) = 1$ *and* $(g_r g_{\alpha_{n-1}}\cdots g_{\alpha_1})(v) = 0$ *for all* $r \in R$.
Our goal is to construct a section χ for Ψ in this case. First observe that $(g_{\beta'}g_{\alpha_{n-1}}\cdots g_{\alpha_1})(v) = 0$ for all $\beta' \in B'$ as well, because $Jpv \subset \sum_r Krpv + J^2pv = J^2pv$ implies $Jpv = 0$.
Define $\chi = (\chi_x) : U \longrightarrow V$ by

$$\begin{aligned} \chi_i(1) &:= (g_{\alpha_{i-1}\cdots\alpha_1})(v) && \text{for } i \in \{1,\dots,n\} \text{ and} \\ \chi_x &:= 0 && \text{for } x \notin \{1,\dots,n\}. \end{aligned}$$

Once we have checked that $\chi \in \mathrm{Hom}_\Lambda(U,V)$, the equality $\Psi_1\chi_1(1) = 1$ will clearly imply $\Psi\chi = \mathrm{id}$, completing the treatment of the first case.
So let us check that χ is Λ-linear. That $g_{\alpha_i}\chi_i = \chi_{i+1} = \chi_{i+1}f_{\alpha_i}$ for $i \in \{1,\ldots,n-1\}$ is clear; moreover, we compute

$$\begin{aligned} g_\delta\chi_{\mathrm{s}(\delta)}(1) &= (g_\delta g_{\alpha_{s(\delta)-1}}\cdots g_{\alpha_1})(v) \\ &= f_\delta(1)(g_{\alpha_{\mathrm{t}(\delta)-1}}\cdots g_{\alpha_1})(v) \\ &= \chi_{\mathrm{t}(\delta)}f_\delta(1). \end{aligned}$$

Now let $\beta \in B \cup B'$. Then $(\beta\alpha_{\mathrm{s}(\beta)-1}\cdots\alpha_1)(v) \in Jpv = 0$, and again $g_\beta\chi_{\mathrm{s}(\beta)} = 0 = \chi_{\mathrm{t}(\beta)}f_\beta$.
Case 2: *For every $v \in V_1$ with $\Psi_1(v) = 1$, there exists $r \in R$ with $(g_r g_{\alpha_{n-1}}\cdots g_{\alpha_1})(v) \neq 0$.*
In this case, we will construct a retraction χ for Φ. First we note that there exist linear maps $\omega_r : V_{u(r)} \longrightarrow K$ for $r \in R$ such that

$$\Psi_1 = \sum_{r\in R}\omega_r g_r g_{\alpha_{n-1}}\cdots g_{\alpha_1}.$$

Define $\chi = (\chi_x) : V \longrightarrow JU$ by

$$\begin{array}{lll} \chi_i & := \Psi_i - \sum_{r\in R}\omega_r g_r g_{\alpha_{n-1}\cdots\alpha_i} & \text{for } i \in \{1,\ldots,n\} \text{ and} \\ \chi_x & := 0 & \text{for } x \notin \{1,\ldots,n\}. \end{array}$$

Again we need to check that χ is Λ-linear. For that purpose, we compute $\chi_1 = 0$,

$$\begin{aligned} f_{\alpha_i}\chi_i &= \Psi_{i+1}g_{\alpha_i} - (\sum_{r\in R}\omega_r g_r g_{\alpha_{n-1}\cdots\alpha_{i+1}})g_{\alpha_i} \\ &= \chi_{i+1}g_{\alpha_i} \end{aligned}$$

for $i \in \{1,\ldots,n-1\}$, and

$$\begin{aligned} f_\delta\chi_{\mathrm{s}(\delta)} &= \Psi_{\mathrm{t}(\delta)}g_\delta - f_\delta(1)\sum_{r\in R}\omega_r g_r g_{\alpha_{n-1}\cdots\alpha_{s(\delta)}} \\ &= \Psi_{\mathrm{t}(\delta)}g_\delta - \sum_{r\in R}\omega_r g_r g_{\alpha_{n-1}\cdots\alpha_{\mathrm{t}(\delta)}}g_\delta \\ &= \chi_{\mathrm{t}(\delta)}g_\delta \end{aligned}$$

for $\delta \in D$. In addition, we obtain $\chi_1 g_{\gamma'} = 0 = f_{\gamma'}\chi_{\mathrm{s}(\gamma')}$ for $\gamma' \in C'$. If $\gamma \in C$, then we can clearly assume that the corresponding element $w \in pJ$ from condition (2)(b)(i) has the form $w = pw'$ with $w' \in e_1Je_{\mathrm{s}(\gamma)}$, and it follows

$$h_{\mathrm{t}(\gamma)}g_\gamma = f_\gamma\Psi_{\mathrm{s}(\gamma)} - \sum_{r\in R}\omega_r g_r g_{\alpha_{n-1}\cdots\alpha_{\mathrm{t}(\gamma)}}g_\gamma$$

$$\begin{aligned}
&= 0 - \sum_{r \in R} \omega_r g_{\alpha_{n-1} \cdots \alpha_{\mathrm{t}(\gamma)}} g_{w'} \\
&= -\Psi_1 g_{w'} \\
&= -f_{w'} \Psi_{\mathrm{s}(\gamma)} \\
&= 0 = f_\gamma \chi_{\mathrm{s}(\gamma)}.
\end{aligned}$$

Hence χ belongs indeed to $\mathrm{Hom}_\Lambda(V, JU)$. That $\chi\Phi = \mathrm{id}_{JU}$ is a consequence of the following computation:

$$\begin{aligned}
\chi_2 \Phi_2(1) &= \Psi_2 \Phi_2(1) - \sum_{r \in R} \omega_r g_r g_{\alpha_{n-1} \cdots \alpha_2} \Phi_2(1) \\
&= 1 - \sum_{r \in R} \omega_r \Phi_{u(r)} f_r f_{\alpha_{n-1} \cdots \alpha_2}(1) \\
&= 1.
\end{aligned}$$

Thus Φ is a split monomorphism in the second case, which shows that the inclusion $JU \longrightarrow U$ cannot be factored nontrivially. $\square$

Proof of "(1) $\Rightarrow$ (2)(a)". Define a relation $<$ on $\mathrm{V}\Gamma$ by setting

$$x < y :\Longleftrightarrow \text{there exists a non-trivial directed path from } x \text{ to } y \text{ in } \Gamma$$

This is in fact a partial order since Γ does not contain oriented cycles.
Now assume that the embedding $JU \longrightarrow U$ is irreducible and let $x_0 \in \mathrm{V}\Gamma$ so that

$$\beta\alpha_{\mathrm{s}(\beta)-1} \cdots \alpha_1 \in Jp \quad \text{for all } \beta \in B \text{ with } \mathrm{t}(\beta) < x_0$$

and

$$\delta\alpha_{\mathrm{s}(\delta)-1} \cdots \alpha_1 \in K\alpha_{\mathrm{t}(\delta)-1} \cdots \alpha_1 \quad \text{for all } \delta \in D \text{ with } \mathrm{t}(\delta) < x_0.$$

We will show that then also

$$\beta\alpha_{\mathrm{s}(\beta)-1} \cdots \alpha_1 \in Jp \quad \text{for all } \beta \in B \text{ with } \mathrm{t}(\beta) = x_0$$

and

$$\delta\alpha_{\mathrm{s}(\delta)-1} \cdots \alpha_1 \in K\alpha_{\mathrm{t}(\delta)-1} \cdots \alpha_1 \quad \text{for all } \delta \in D \text{ with } \mathrm{t}(\delta) = x_0,$$

which will prove our claim.
Case 1: $x_0 \notin \{1, \ldots, n\}$.
Set $\hat{B} := \{\beta \in B \mid \mathrm{t}(\beta) = x_0\}$. If, contrary to our claim, there exists $\hat{\beta} \in \hat{B}$ with $\hat{\beta}\alpha_{\mathrm{s}(\hat{\beta})-1} \cdots \alpha_1 \notin Jp$, then we can find a K-linear map $\phi : e_{x_0}\Lambda e_1 \longrightarrow K$

with $Jp \subset \operatorname{kern}(\phi)$ and $\phi(\hat{\beta}\alpha_{\mathrm{s}(\hat{\beta})-1}\cdots\alpha_1) \neq 0$. This will allow us to define the following representation $V = ((V_x),(g_\alpha))$ of Γ:

$$V_x = \begin{cases} K, & \text{if } x \in \{1, n, x_0\}; \\ K \oplus K, & \text{if } x \in \{2, \dots, n-1\}; \\ 0, & \text{else} \end{cases}$$

$$g_\alpha = \begin{cases} f_\alpha, & \text{if } \mathrm{s}(\alpha) = 1 \text{ and } \mathrm{t}(\alpha) = n; \\ \binom{0}{f_\alpha}, & \text{if } \mathrm{s}(\alpha) = 1 \text{ and } \mathrm{t}(\alpha) \in \{2, \dots, n-1\}; \\ f_\alpha \oplus f_\alpha, & \text{if } \{\mathrm{s}(\alpha), \mathrm{t}(\alpha)\} \subset \{2, \dots, n-1\}; \\ (f_\alpha, f_\alpha), & \text{if } \mathrm{s}(\alpha) \in \{2, \dots, n-1\} \text{ and } \mathrm{t}(\alpha) = n; \\ \phi(\alpha), & \text{if } \alpha \in \hat{B} \text{ and } \mathrm{s}(\alpha) = 1; \\ (0, \phi(\alpha\alpha_{\mathrm{s}(\alpha)-1}\cdots\alpha_1)), & \text{if } \alpha \in \hat{B} \text{ and } \mathrm{s}(\alpha) > 1; \\ 0, & \text{else.} \end{cases}$$

We will see that V does indeed define a left Λ-module and that the embedding $JU \longrightarrow U$ can be non-trivially factored through V. This violates condition (1).

First of all, we check that V satisfies the relations in I: let $x, y \in \mathrm{V}\Gamma$ and $\mu = \sum_{q \in Q} \mu_q q \in e_x I e_y$ be a relation with $\mu_q \in K$ and a set Q of paths having lengths ≥ 2, starting in y, and terminating in x. Obviously, $g_\mu = 0$ if $y \notin \{1, \dots, n-1\}$ or $x \notin \{1, \dots, n, x_0\}$. If $y \in \{1, \dots, n-1\}$ and $x \in \{1, \dots, n\}$, then $f_\mu = 0$, and since g_μ is one of f_μ, $\binom{0}{f_\mu}$, $f_\mu \oplus f_\mu$, or (f_μ, f_μ), we have $g_\mu = 0$ as well.

Now assume $x = x_0$ and $y \in \{2, \dots, n-1\}$. Since $f_\beta = 0$ for all $\beta \in B$, we have $g_q(1,0) = 0$ for all $q \in Q$ and hence $g_\mu(1,0) = 0$. On the other hand, if $\hat{Q}$ is the set of all those $q \in Q$ which are contingent to certain vertices in $\{y, y+1, \dots, n-1, x_0\}$ but not to any others, we know that every $q \in \hat{Q}$ ends in a certain arrow $\beta(q) \in \hat{B}$, and the second part of the induction hypothesis yields

$$q\alpha_{y-1}\cdots\alpha_1 = k_q \beta(q) \alpha_{\mathrm{s}(\beta(q))-1}\cdots\alpha_1$$

for some scalar $k_q \in K$. Using the first part of the induction hypothesis, we infer that

$$\sum_{q \in Q \setminus \hat{Q}} \mu_q q \alpha_{y-1}\cdots\alpha_1 \in Jp$$

and hence

$$\sum_{q \in \hat{Q}} \mu_q \phi(q\alpha_{y-1}\cdots\alpha_1) = 0.$$

It follows that

$$\begin{aligned} g_\mu(0, f_{\alpha_{y-1}\cdots\alpha_1}(1)) &= \sum_{q\in\hat{Q}} \mu_q g_{\beta(q)}(0, k_q f_{\alpha_{s(\beta(q))-1}\cdots\alpha_1}(1)) \\ &= \sum_{q\in\hat{Q}} \mu_q k_q \phi(\beta(q)\alpha_{\mathrm{s}(\beta(q))-1}\cdots\alpha_1) \\ &= 0. \end{aligned}$$

We thus get $g_\mu = 0$ here. Since the case $x = x_0, y = 1$ can be dealt with analogously, we conclude that V is indeed a Λ-module.

We now define homomorphisms

$$JU \xrightarrow{\Phi=(\Phi_x)} V \xrightarrow{\Psi=(\Psi_x)} U$$

by setting

$$\Phi_x = \begin{cases} 1, & \text{if } x = n; \\ \binom{1}{0}, & \text{if } x \in \{2, \dots, n-1\}; \\ 0, & \text{else} \end{cases}$$

and

$$\Psi_x = \begin{cases} 1, & \text{if } x \in \{1, n\}; \\ (1,1), & \text{if } x \in \{2, \dots, n-1\}; \\ 0, & \text{else.} \end{cases}$$

Obviously, $\Psi\Phi$ equals the embedding $JU \longrightarrow U$. Every splitting $\chi = (\chi_x) : V \longrightarrow JU$ of Φ would have to satisfy $\chi_1 = 0$ and hence $\chi_n = 0$, which is impossible. Every splitting $\chi = (\chi_x) : U \longrightarrow V$ of Ψ would satisfy $\chi_1(1) = 1$ and hence $g_{\hat{\beta}}\chi_{\mathrm{s}(\hat{\beta})}(1) \neq 0$, contradicting the fact that $\chi_{x_0} = 0$. We conclude that $JU \longrightarrow U$ is reducible, a contradiction. Finally note that the second part of (2)(a) is void if $x_0 \notin \{1, \dots, n\}$.

Case 2: $x_0 \in \{1, \dots, n\}$.

Since $x_0 = 1$ is trivial, we assume $x_0 \geq 2$. Note that the first statement of condition (2)(a) is void in this case. Again we will construct a factorization $\Psi\Phi$ of the inclusion $JU \longrightarrow U$ such that Φ does not split. Irreducibility of the inclusion will then force Ψ to split, and this, in turn, will imply that each arrow $\delta \in D$ with $\mathrm{t}(\delta) = x_0$ satisfies the second condition under (2)(a).

Define the following representation $V = ((V_x), (g_\alpha))$ of Γ:

$$V_x = \begin{cases} K, & \text{if } x = 1 \text{ or } x \in \{x_0+1, \dots, n\}; \\ K \oplus K, & \text{if } x \in \{2, \dots, x_0 - 1\}; \\ e_{x_0}\Lambda e_1, & \text{if } x = x_0; \\ 0, & \text{else.} \end{cases}$$

$$g_\alpha = \begin{cases} f_\alpha, & \text{if } \mathrm{s}(\alpha) \in \{1\} \cup \{x_0+1, \dots, n\} \\ & \qquad \text{and } \mathrm{t}(\alpha) \in \{x_0+1, \dots, n\}; \\ \binom{0}{f_\alpha}, & \text{if } \mathrm{s}(\alpha) = 1 \text{ and } \mathrm{t}(\alpha) \in \{2, \dots, x_0 - 1\}; \\ \alpha, & \text{if } \mathrm{s}(\alpha) = 1 \text{ and } \mathrm{t}(\alpha) = x_0; \\ f_\alpha \oplus f_\alpha, & \text{if } \{\mathrm{s}(\alpha), \mathrm{t}(\alpha)\} \subset \{2, \dots, x_0 - 1\}; \\ (f_\alpha(1)\alpha_{x_0-1}\cdots\alpha_1, \alpha\alpha_{s(\alpha)-1}\cdots\alpha_1), & \\ & \text{if } \mathrm{s}(\alpha) \in \{2, \dots, x_0 - 1\} \text{ and } \mathrm{t}(\alpha) = x_0; \\ F_\alpha, & \text{if } \mathrm{s}(\alpha) = x_0 \text{ and } \mathrm{t}(\alpha) \in \{x_0+1, \dots, n\}; \\ (f_\alpha, f_\alpha), & \text{if } \mathrm{s}(\alpha) \in \{2, \dots, x_0 - 1\} \text{ and } \mathrm{t}(\alpha) \in \{x_0+1, \dots, n\}; \\ 0, & \text{else.} \end{cases}$$

Here, $F_\alpha : e_{x_0}\Lambda e_1 \longrightarrow K$ is defined by $F_\alpha(w) = f_{\alpha w}(1)$. Again, we have to check that V actually yields a Λ-module: let $\mu = \sum_{q \in Q} \mu_q q \in e_x I e_y$ be a relation with $\mu_q \in K$, where Q is a set of paths of length ≥ 2 starting in $y \in \mathrm{V}\Gamma$ and terminating in $x \in \mathrm{V}\Gamma$.

It is clear that $g_\mu = 0$ whenever $\{x, y\} \not\subset \{1, \dots, n\}$. Moreover, if $\alpha, \beta \in \mathrm{A}\Gamma$ with $\mathrm{t}(\alpha) = \mathrm{s}(\beta) = x_0$ and $\{\mathrm{s}(\alpha), \mathrm{t}(\beta)\} \subset \{1, \dots, n\}$, then we have $f_{\beta\alpha}(1) = F_\beta(\alpha\alpha_{s(\alpha)-1}\cdots\alpha_1) = F_\beta(f_\alpha(1)\alpha_{x_0-1}\cdots\alpha_1)$, which shows that $g_{\beta\alpha}$ is either $f_{\beta\alpha}$ or $(f_{\beta\alpha}, f_{\beta\alpha})$. Hence $g_\mu = 0$ if $x_0 \notin \{x, y\}$.

If $y = x_0$, we get $g_\mu(w) = f_{\mu w}(1) = 0$ for all $w \in e_x\Lambda e_1$.

The remaining case is $x = x_0$. We will explicitly deal only with the situation $y \geq 2$ since the case $y = 1$ is analogous. First we compute $g_\mu(1, 0) = f_\mu(1)\alpha_{x_0-1}\cdots\alpha_1 = 0$. Moreover, we can write

$$q\alpha_{y-1}\cdots\alpha_1 = k_q\delta(q)\alpha_{\mathrm{s}(\delta(q))-1}\cdots\alpha_1$$

with certain $k_q \in K$ and $\delta(q) \in D$ for every $q \in Q$ (note that all those paths $q \in Q$ that involve vertices other than $\{1, \dots, n\}$ are contained in the ideal I by induction hypothesis, since Γ does not contain oriented cycles). It then follows that

$$\begin{aligned} g_\mu(0, f_{\alpha_{y-1}\cdots\alpha_1}(1)) &= \sum_{q \in Q} \mu_q g_{\delta(q)}(0, k_q f_{\alpha_{s(\delta(q))-1}\cdots\alpha_1}(1)) \\ &= \sum_{q \in Q} \mu_q k_q \delta(q)\alpha_{\mathrm{s}(\delta(q))-1}\cdots\alpha_1 \end{aligned}$$

$$= 0.$$

Hence V is indeed a Λ-module. To define homomorphisms

$$JU \xrightarrow{\Phi=(\Phi_x)} V \xrightarrow{\Psi=(\Psi_x)} U\ ,$$

we set

$$\Phi_x = \begin{cases} \binom{1}{0}, & \text{if } x \in \{2, \dots, x_0 - 1\}; \\ \alpha_{x_0-1} \cdots \alpha_1, & \text{if } x = x_0; \\ 1, & \text{if } x \in \{x_0 + 1, \dots, n\}; \\ 0, & \text{else} \end{cases}$$

and

$$\Psi_x = \begin{cases} 1, & \text{if } x = 1 \text{ or } x \in \{x_0 + 1, \dots, n\}; \\ (1, 1), & \text{if } x \in \{2, \dots, x_0 - 1\}; \\ F, & \text{if } x = x_0; \\ 0, & \text{else.} \end{cases}$$

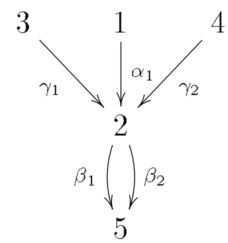
Here $F : e_{x_0}\Lambda e_1 \longrightarrow K$ is given by $F(w) = f_w(1)$. It is easy to see that these are indeed Λ-homomorphisms which yield a factorization of the radical embedding of U. As seen before, Φ never splits. Hence Ψ must split; let $\chi = (\chi_x) : U \longrightarrow V$ be a splitting. For every $\delta \in D$ with $\mathrm{t}(\delta) = x_0$ we use the equality $\chi_1(1) = 1 = \chi_{\mathrm{s}(\delta)}(1)$ to obtain

$$\begin{aligned} \delta\alpha_{\mathrm{s}(\delta)-1} \cdots \alpha_1 &= g_\delta(1) \\ &= g_\delta \chi_{\mathrm{s}(\delta)}(1) \\ &= \chi_{x_0} f_{\delta\alpha_{\mathrm{s}(\delta)-1}\cdots\alpha_1}(1) \\ &= f_{\delta\alpha_{\mathrm{s}(\delta)-1}\cdots\alpha_1}(1) \chi_{\mathrm{t}(\delta)} f_{\alpha_{\mathrm{t}(\delta)-1}\cdots\alpha_1}(1) \\ &= f_{\delta\alpha_{\mathrm{s}(\delta)-1}\cdots\alpha_1}(1) \alpha_{\mathrm{t}(\delta)-1} \cdots \alpha_1, \end{aligned}$$

which is what we claimed. □

EXAMPLES. In order to provide a better understanding of the different cases that would have to be dealt with in a proof of "(1) ⇒ (2)(b)", we include here a series of examples where condition (2)(b) of the Conjecture is violated. A non-trivial factorization of the radical embedding is given in each of these cases.

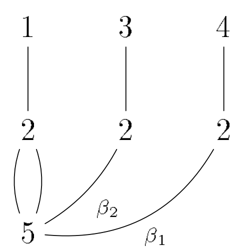
(a) Suppose Γ is given by



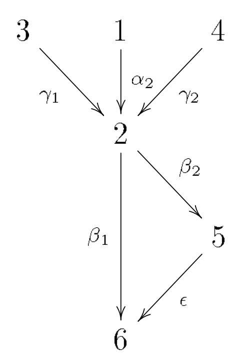
and the relations are

$$\beta_1\alpha_1 = \beta_2\alpha_1 \quad \text{and} \quad \beta_1\gamma_1 = 0 = \beta_2\gamma_2.$$

Here U is the unique uniserial with mast α_1. The embedding $JU \longrightarrow U$ can then be factored non-trivially through a module with graph



(b) Now Γ is given by



with relations

$$\epsilon\beta_2\gamma_2 = \beta_1\gamma_2 \quad \text{and} \quad \beta_1\gamma_1 = 0 = \beta_2\alpha_1.$$

Again, U is the unique uniserial with mast α_1. In this case, the radical embedding can be factored through the indecomposable with graph

$$\begin{tikzcd}
1 \arrow[d, dash] & 4 \arrow[d, dash] & 3 \arrow[d, dash] \\
2 \arrow[ddd, dash] & 2 \arrow[d, dash] \arrow[dddl, dash] & 2 \arrow[dl, dash] \\
 & 5 \arrow[ddl, dash] & \\
 & & \\
6 & &
\end{tikzcd}$$

(c) Consider the quiver Γ

$$\begin{tikzcd}
1 \arrow[d, "\alpha_1"'] & 4 \arrow[dl, "\gamma_1"] \\
2 \arrow[d, "\alpha_2"'] \arrow[dd, bend left, "\beta_2"] & \\
3 \arrow[d, "\beta_1"'] & \\
5 &
\end{tikzcd}$$

with relations

$$\beta_2\alpha_1 = \beta_1\alpha_2\alpha_1 \quad \text{and} \quad \beta_2\gamma_1 = 0.$$

The radical embedding of the uniserial with mast $\alpha_2\alpha_1$ can be factored through the following indecomposable module:

$$\begin{tikzcd}
1 \arrow[d, dash] & & 4 \arrow[d, dash] \\
2 \arrow[d, dash] \arrow[dd, dash, bend left] & 2 \arrow[d, dash] \arrow[ddl, dash] & 2 \arrow[dl, dash] \\
3 \arrow[d, dash] & 3 \arrow[dl, dash] & \\
5 & &
\end{tikzcd}$$

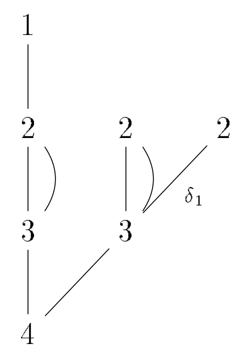
(d) In our final example, let Γ be given by

$$\begin{tikzcd} 1 \arrow[d, "\alpha_1"'] \\ 2 \arrow[d, "\alpha_2"'] \arrow[d, bend left, "\delta_1"] \\ 3 \arrow[d, "\beta_1"'] \\ 4 \end{tikzcd}$$

and consider the relation

$$\delta_1\alpha_1 = \alpha_2\alpha_1.$$

We can factor the radical embedding of the uniserial with mast $\alpha_2\alpha_1$ through the module



REMARK. In order to tackle the remaining implication "(1) $\Rightarrow$ (2)(b)" of the conjecture, it may be convenient to have the following reformulation of condition (2)(b) at hand:
(2)(b') *There exists a family* $(w_\gamma) \in (pJ)^C$, *such that for every* $x \in \mathrm{V}\Gamma$ *and* $\mu \in e_xJp/e_xJ^2p$, *we can find* $r \in e_xJe_n$ *with* $\mu = rp + e_xJ^2p$ *and* $r\alpha_{n-1}\cdots\alpha_{\mathrm{t}(\gamma)}\gamma = rw_\gamma$ *for all* $\gamma \in C$ *and* $r\alpha_{n-1}\cdots\alpha_{\mathrm{t}(\delta)}\delta \in Kr\alpha_{n-1}\cdots\alpha_{\mathrm{s}(\delta)}$ *for all* $\delta \in D$.

Assume that condition (1) holds, i. e., that the canonical embedding $JU \longrightarrow U$ is irreducible, and that (2)(b') is violated. We then get, for every family (w_γ), a special vertex x and an element $\mu \in e_xJp/e_xJ^2p$ from the negation of this statement. Since (2)(a) holds, this allows us to "lengthen" U to a uniserial module $\hat{U}$ in such a fashion that U is an epimorphic image of $\hat{U}$ and $\operatorname{soc}\hat{U} \simeq \Lambda e_x/Je_x$ (note however that there is a choice involved: $\hat{U}$ is not uniquely determined by U and μ). Here are two potential approaches to the construction of a module M through which the radical embedding of U factors non-trivially:

(a) Let M be the module obtained from gluing the socles of $\hat{U}$ and $\mathrm{D}(e(x)\Lambda)$ (where $\mathrm{D} = \mathrm{Hom}_K(-, K)$ denotes the usual duality). The problem then is to find a "good" map from JU to M.
(b) This time, we begin by gluing the socles of $\hat{U}$ and $J\hat{U}$ together to obtain $\check{M}$; this allows for a natural embedding of JU. Of course, this particular embedding splits, and we have to extend $\check{M}$ to a module M having $\check{M}$ as an epimorphic image in order to prevent this from happening.

1.3. Triangles of Uniserials

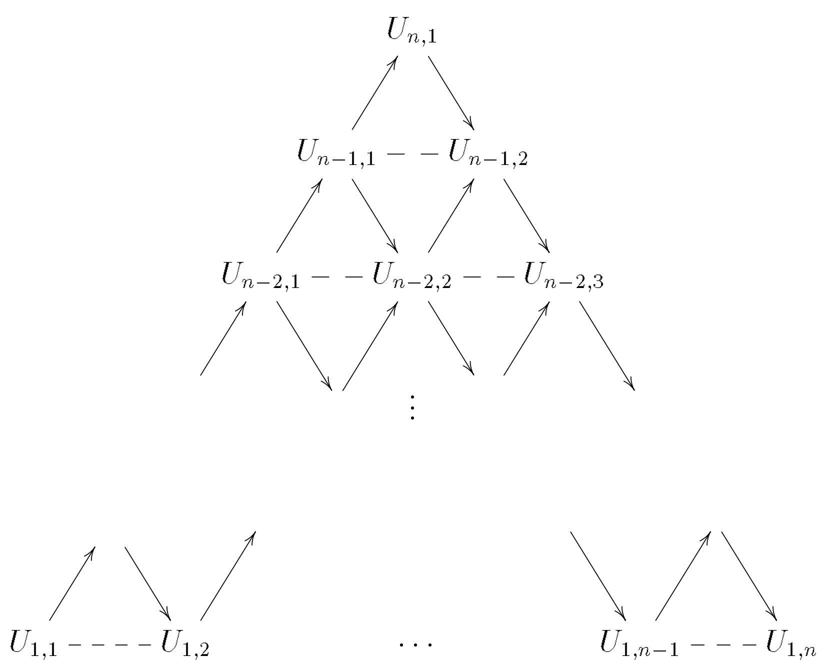
In this section, we define triangles of uniserials and their tips, and then classify all those triangles over triangular algebras in combinatorial terms on the basis of quiver and relations.

The prototype of a triangle of uniserials is given by the Auslander-Reiten-quiver of the path algebra of a linearly oriented diagram $\mathbb{A}_n$, i. e. of the algebra $K\Gamma$ where Γ is given by

$$1 \longrightarrow 2 \longrightarrow \cdots \longrightarrow n.$$

In general:

Definition 1.3.1. A *triangle of uniserial* Λ*-modules* is a full translation subquiver of the Auslander-Reiten-quiver of Λ having the following form:



where $n \in \mathbb{N}$ and the U_{ij} are pairwise non-isomorphic uniserial left Λ-modules with

$$\mathrm{length}(U_{ij}) = i \quad \text{for all } i \text{ and } j.$$

The module $U_{n,1}$ is called the *tip* of this triangle; the triangle is called *maximal* if it is not properly contained in a bigger triangle of uniserials.

Note that the length restrictions imply that the maps $U_{i,j} \longrightarrow U_{i+1,j}$ are (isomorphic to) radical embeddings and the maps $U_{i,j} \longrightarrow U_{i-1,j+1}$ are (isomorphic to) socle factor projections, compare 1.1.3. Moreover, if Λ is triangular, then the requirement that the U_{ij} be pairwise non-isomorphic is actually a consequence of the rest of the definition: the U_{ij} clearly have different sequences of composition factors. Furthermore, every triangle of uniserials contains at least one simple module, which implies that there can only be finitely many different triangles. The simple modules are precisely the triangles consisting of a single module.

Continuing to employ the notation introduced at the beginning of the previous section, we will use that section's results in the following form:

PROPOSITION 1.3.2. (I) *If the natural embedding $JU \longrightarrow U$ is irreducible, then*

$$\beta\alpha_{\mathrm{s}(\beta)-1}\cdots\alpha_1 \in Jp \quad \textit{for every } \beta \in B$$

and

$$\delta\alpha_{\mathrm{s}(\delta)-1}\cdots\alpha_1 \in K\alpha_{\mathrm{t}(\delta)-1}\cdots\alpha_1 \quad \textit{for every } \delta \in D.$$

(II) *If $B = C = D = \varnothing$ and U is not simple, then $JU \longrightarrow U$ is irreducible.*
(III) *If the natural epimorphism $U \longrightarrow U/\operatorname{soc} U$ is irreducible, then*

$$\alpha_{n-1}\cdots\alpha_{\mathrm{t}(\gamma)}\gamma \in pJ \quad \textit{for every } \gamma \in C$$

and

$$\alpha_{n-1}\cdots\alpha_{\mathrm{t}(\delta)}\delta \in K\alpha_{n-1}\cdots\alpha_{\mathrm{s}(\delta)} \quad \textit{for every } \delta \in D.$$

(IV) *If $B = C = D = \varnothing$ and U is not simple, then $U \longrightarrow U/\operatorname{soc} U$ is irreducible.*

PROOF. Part (I) was proved as "(1) $\Rightarrow$ (2)(a)" of Conjecture 1.2.1 in the last section while part (II) is a trivial consequence of the same conjecture's "(2) $\Rightarrow$ (1)", which is also known to be true. The parts (III) and (IV) follow by duality: if $\mathrm{D} : \Lambda\text{-}\mathrm{mod} \longrightarrow \Lambda^{\mathrm{op}}\text{-}\mathrm{mod}$ denotes the usual duality $\mathrm{D} = \mathrm{Hom}_K(-, K)$, then the dual of the natural epimorphism $U \longrightarrow U/\operatorname{soc} U$ is just the natural embedding $\operatorname{rad} \mathrm{D}U \longrightarrow \mathrm{D}U$; one of these maps is irreducible if and only if the other is. The results follow now from (I) and (II) using the fact that the quiver of Λ^{op} is Γ^{op}, that the inverse path to p can be chosen as mast for $\mathrm{D}U$, and that the Jacobson radicals of Λ and Λ^{op} coincide. $\square$

The following lemma provides the missing link between triangles of uniserials and radical embeddings resp. socle factor projections.

LEMMA 1.3.3. *If $B = C = D = \varnothing$, then the uniserial module U appears in a triangle of uniserials.*

PROOF. First note that the condition on U is obviously inherited by all subfactors of U; we thus have only to show that the short exact sequence

$$\begin{tikzcd}[column sep=small]
 & & U \arrow[drr] & & & \\
0 \arrow[r] & JU \arrow[ur] \arrow[dr] & \oplus & & U/\operatorname{soc} U \arrow[r] & 0 \\
 & & JU/\operatorname{soc}(JU) \simeq J(U/\operatorname{soc} U) \arrow[urr] & & &
\end{tikzcd}$$

is almost split, since we know from Proposition 1.3.2 that all appearing maps are irreducible or 0. To this end, it clearly suffices to prove that $\tau(U/\operatorname{soc} U) = JU$. Using $B = C = D = \varnothing$, we see that a projective resolution of $U/\operatorname{soc} U$ is given by

$$\Lambda e_n \xrightarrow{-\cdot p} \Lambda e_1 \longrightarrow U/\operatorname{soc} U \longrightarrow 0$$

and that

$$0 \longrightarrow JU \longrightarrow \mathrm{D}(e_n\Lambda) \xrightarrow{\mathrm{D}(p\cdot -)} \mathrm{D}(e_1\Lambda)$$

is an injective resolution of JU. However, the map $\mathrm{D}(p \cdot -)$ is just $\mathcal{N}(- \cdot p)$ (with the Nakayama functor $\mathcal{N} = \mathrm{D}\operatorname{Hom}_\Lambda(-, \Lambda)$), the kernel of which is $\tau(U/\operatorname{soc} U)$. $\square$

We are now in a position to prove the announced characterization of triangles of uniserials and their tips. We say that an arrow $\alpha \in \Lambda\Gamma$ *leaves the path* $p = 1 \xrightarrow{\alpha_1} 2 \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_{n-1}} n$ if $\mathrm{s}(\alpha) \in \{1, \ldots, n\}$ and $\alpha \notin \{\alpha_1, \ldots, \alpha_{n-1}\}$. The arrow α *enters* p if $\mathrm{t}(\alpha) \in \{1, \ldots, n\}$ and $\alpha \notin \{\alpha_1, \ldots, \alpha_{n-1}\}$. Moreover, we say that the path p is *maximal* with property P if p satisfies P, and whenever q is a path such that p is a proper subpath of q, then q does not satisfy P.

THEOREM 1.3.4. *The uniserial module U with mast p appears in a triangle of uniserials if and only if every arrow leaving p leaves at the terminal vertex of p and every arrow entering p enters at the starting vertex of p. The module U is the tip of a maximal triangle of uniserials if and only if the mast p of U is maximal with the property that p is not contained in the ideal of relations I and every leaving arrow leaves at the terminal vertex and every entering arrow enters at the starting vertex. Every triangle of uniserials is contained in (at least one) maximal triangle.*

Proof. Returning to the notation introduced in the last section, the property that every arrow leaving p leaves at the terminal vertex of p and every arrow entering p enters at the starting vertex of p is clearly equivalent to the statement $B = C = D = \varnothing$.

First assume that U appears in a triangle of uniserials. Then so do all its subfactors. Hence it suffices to prove that there cannot exist any $\beta \in B$ with $\mathrm{s}(\beta) = 1$, any $\gamma \in C$ with $\mathrm{t}(\gamma) = n$, nor any $\delta \in D$ with $\mathrm{s}(\delta) = 1$ and $\mathrm{t}(\delta) = n$. We have already noted that, in case U is not simple, both the radical embedding and the socle factor projection of U are irreducible. Taking this into account, Proposition 1.3.2 does the job since none of the relations in I defining Λ can contain a single arrow.

Clearly, if U appears inside a triangle of uniserials but not as its tip, then the mast of the uniserial sitting "over" U is a proper prolongation of p still satisfying $B = C = D = 0$, and p can not have been maximal.

Now assume that p is not maximal. Then we can find a path $p' = \alpha p$ (or $p' = p\alpha$) with $\alpha \in \mathrm{A}\Gamma$ such that, if B', C', and D' denote the corresponding sets of arrows, we have $B' = C' = D' = \varnothing$. This latter condition, together with $p \notin I$, implies that there exists a uniserial left Λ-module U' with mast p' and $JU' = U$ (or $U'/\operatorname{soc} U' = U$). Obviously, U' is the tip of a triangle of uniserials which properly contains U.

If p satisfies $B = C = D = \varnothing$, then there exists a path p'' (with corresponding sets of arrows B'', C'', D'') that contains p as a subpath and is maximal with the properties $B'' = C'' = D'' = \varnothing$ and $p'' \notin I$. The path p'' is then the mast of the tip of a maximal triangle of uniserials containing U. $\square$

We will see in 2.3.3 that, over a hereditary algebra, every finitely generated uniserial module U with the property that both the radical embedding $JU \longrightarrow U$ and the socle factor projection $U \longrightarrow U/\operatorname{soc} U$ are irreducible is the tip of a triangle of uniserials. This is false over triangular algebras, and armed with the results we have, we can easily construct a counterexample:

Example. Consider the quiver Γ

$$\begin{tikzcd}
1 \arrow[d, "\alpha_1"] \\
2 \arrow[d, "\alpha_2"] \arrow[d, bend left=60, "\delta"] \\
3 \arrow[d, "\alpha_3"] \\
4
\end{tikzcd}$$

and the relations

$$\alpha_2\alpha_1 = \delta\alpha_1 \quad \text{and} \quad \alpha_3\alpha_2 = \alpha_3\delta.$$

If U is the unique uniserial module with mast $\alpha_3\alpha_2\alpha_1$, then $JU \longrightarrow U$ and $U \longrightarrow U/\operatorname{soc} U$ are both irreducible since condition (2) (and its dual) of Conjecture 1.2.1 are satisfied. However, the maps $J^2U \longrightarrow JU$ and $U/\operatorname{soc} U \longrightarrow U/\operatorname{soc}^{(2)} U$ are not irreducible because condition (2)(a) (resp. its dual) are violated. Furthermore, both the radical embedding of $U/\operatorname{soc} U$ and the socle factor projection of JU are irreducible, while the radical embedding and socle factor projection of $JU/\operatorname{soc}(JU)$ are not. Moreover, it is straightforward to check that $\tau(U/\operatorname{soc} U) = JU$. Therefore, the "degenerate triangle of uniserials with tip U" looks like this:

$$\begin{tikzcd}
 & U \arrow[dr] & \\
JU \arrow[ur] \arrow[dr] \arrow[rr, dash, dashed] & & U/\operatorname{soc} U \\
 & JU/\operatorname{soc}(JU) \arrow[ur] &
\end{tikzcd}$$

CHAPTER 2

Uniserial Modules over Hereditary Algebras

2.1. Notation and First Results

This chapter contains our results concerning finitely generated uniserial left modules over split hereditary finite-dimensional algebras. The first section provides the notational framework and presents some elementary facts. The following section examines those uniserials which have uniserial Auslander-Reiten translate, and the third section exploits the results of the previous chapter to provide a full understanding of triangles of uniserials in the hereditary situation.

Throughout this chapter, Γ will be a finite quiver without oriented cycles, K will be a field, and $\Lambda = K\Gamma$ will be the split hereditary finite-dimensional path algebra defined by Γ.

We call a path p' a *right subpath* of the path p if there exists a path r with $p = rp'$. Left subpaths are defined similarly. Two paths p_1, p_2 are called *comparable* if one is a left or right subpath of the other.

Following [**14**], if p is a path in Γ, a *detour on p* is defined to be a tuple (α, p') where $\alpha \in A\Gamma$ and p' is a right subpath of p with $\mathrm{t}(p') = \mathrm{s}(\alpha)$ and such that $\alpha p'$ is not a right subpath of p. Since our quiver does not have any oriented cycles, the detour is uniquely determined by α. By abuse of language, we also call α itself a detour on p and write "$\alpha \wr p$" for this situation.

Now let a scalar $\lambda_\alpha \in K$ be given for every detour $\alpha \wr p$. Define $I(p, (\lambda_\alpha)_{\alpha\wr p})$ to be the left Λ-submodule of Λe_1 generated by the elements $\alpha\alpha_{\mathrm{s}(\alpha)-1}\cdots\alpha_1 - \lambda_\alpha\alpha_{\mathrm{t}(\alpha)-1}\cdots\alpha_1$ for $\alpha \wr p$ and $\beta\alpha_{\mathrm{s}(\beta)-1}\cdots\alpha_1$ for all non-detours β that leave p. We can then define a left Λ-module $U(p, (\lambda_\alpha)_{\alpha\wr p})$ by setting

$$U(p, (\lambda_\alpha)_{\alpha\wr p}) := \Lambda e_1 / I(p, (\lambda_\alpha)_{\alpha\wr p}).$$

We write $U(p)$ instead of $U(p, (\lambda_\alpha))$ if $\lambda_\alpha = 0$ for every $\alpha \wr p$ (in particular if p does not admit any detours whatsoever). The module $U(p, (\lambda_\alpha))$ is uniserial with mast p, and every uniserial Λ-module with mast p is isomorphic to a $U(p, (\lambda_\alpha))$ for a suitable family of scalars $(\lambda_\alpha)_{\alpha\wr p}$. Moreover,

$$U(p, (\lambda_\alpha)) \simeq U(p, (\mu_\alpha)) \iff (\lambda_\alpha) = (\mu_\alpha).$$

This means that the varieties V_p introduced in [**14**] in order to parameterize uniserials are full affine spaces in the hereditary case and provide a 1-1 parameterization.

In general, the problem of classifying those finite-dimensional algebras having only finitely many finitely generated uniserial left modules is very complicated, compare [**15**]. In the hereditary situation however, it is trivial:

LEMMA 2.1.1. *The finite-dimensional hereditary algebra $\Lambda = K\Gamma$ admits infinitely many non-isomorphic uniserial left modules if and only if $|K| = \infty$ and Γ contains a subquiver of the form*

PROOF. If Λ admits infinitely many uniserial modules, then there must be one path p that serves as mast of infinitely many uniserials, since there are only finitely many paths. By the remarks preceding the lemma, we must then have infinitely many different families $(\lambda_\alpha)_{\alpha \wr p}$, which implies that K is infinite and that there exists at least one detour on p.

On the other hand, if a path p with a detour α exists and K is infinite, we can concoct infinitely many different families $(\lambda_\alpha)_{\alpha \wr p}$ yielding infinitely many non-isomorphic uniserial modules. $\square$

2.2. The Auslander-Reiten Translation

This section contains the classification of those finitely generated uniserial modules U over a split hereditary finite-dimensional algebra which have a uniserial Auslander-Reiten translate τU. We will explicitly describe the Auslander-Reiten translate, using the notation developed in the preceding section.

Initially, certain examples suggested to us that, over hereditary algebras of finite type, the uniserials might be "connected to projectives or injectives via uniserials"; by which we mean that for every uniserial U, either all the modules $\tau^i U$ $(i \in \mathbb{N})$ are uniserial or undefined or else all the modules $\tau^{-i} U$ $(i \in \mathbb{N})$, are uniserial or undefined. Closer inspection proved this conjecture to be false; we managed to classify all those hereditary algebras of finite type for which it does hold, and this classification will conclude the section.

PROPOSITION 2.2.1. *Let $p = 1 \xrightarrow{\alpha_1} 2 \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_{n-1}} n$ be a path in Γ and $\lambda_\alpha \in K$ for every $\alpha \wr p$. Write $U := U(p, (\lambda_\alpha))$. Then τU is defined and uniserial if and only if the following two conditions are both satisfied:*

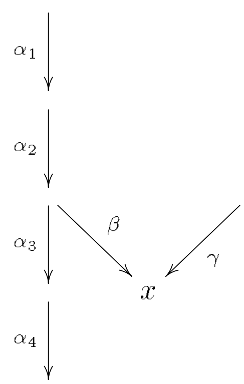
(1) *There is precisely one arrow β leaving p.*
(2) *Writing $\beta : i \longrightarrow x$, all paths ending in x and not having $\beta\alpha_{i-1}\cdots\alpha_1$ as left subpath are comparable.*

If this is the case and if p' is the longest path ending in x and not having $\beta\alpha_{i-1}\cdots\alpha_1$ as left subpath, then p' has the same detours as p (in fact, there can only be at most one detour), and we have

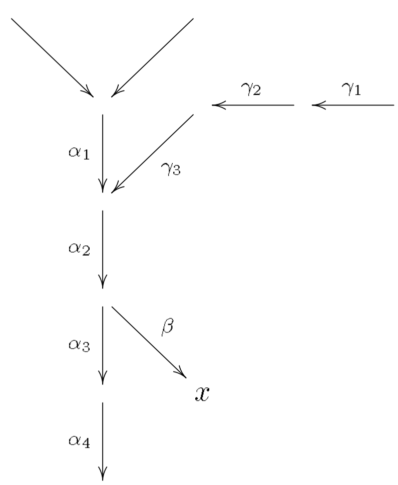
$$\tau U = U(p', (\lambda_\alpha)).$$

Before we prove the proposition, we give some examples to illustrate the condition appearing in the statement:

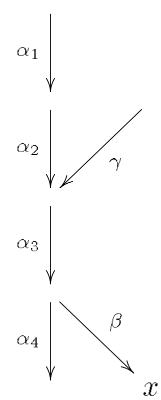
EXAMPLES. (a) Consider the quiver Γ



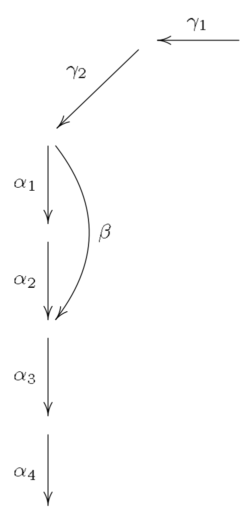
If U is the unique uniserial with mast $p = \alpha_4\alpha_3\alpha_2\alpha_1$, then $\tau(U(p))$ is not uniserial.
(b) If Γ is given by



and U is again the uniserial with mast $p = \alpha_4\alpha_3\alpha_2\alpha_1$, then $\tau U = U(\beta\alpha_2\gamma_3\gamma_2\gamma_1)$. We see that, in general, the lengths of U and τU can be arbitrarily far apart.
(c) The module $\tau(U(p))$ is not uniserial in the situation

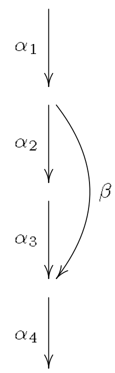


(d) In the case



where, again, $p = \alpha_4\alpha_3\alpha_2\alpha_1$, we have $\tau(U(p, \lambda_\beta)) = U(\alpha_2\alpha_1\gamma_2\gamma_1, \lambda_\beta)$. Here we get a whole family of Auslander-Reiten sequences with uniserial end terms, indexed by elements from the base field K.

(e) Finally, if Γ is given by



then $\tau(U(p,\lambda_\beta))$ is not uniserial.

Proof of the Proposition. Let $\mathrm{D} = \mathrm{Hom}_K(-,K)$ denote the usual duality and let $\mathcal{N} = \mathrm{D}\,\mathrm{Hom}_\Lambda(-,\Lambda)$ be the Nakayama functor. For $x \in e_i\Lambda e_j$, we have $\mathcal{N}(\ \Lambda e_i \xrightarrow{-\cdot x} \Lambda e_j\ ) = \mathrm{D}(e_i\Lambda) \xrightarrow{\mathrm{D}(x\cdot -)} \mathrm{D}(e_j\Lambda)$. Furthermore, if $M \in \Lambda\text{-}\operatorname{mod}$ is indecomposable and non-projective and $P_1 \xrightarrow{\phi} P_0 \longrightarrow M \longrightarrow 0$ is a minimal projective resolution, then $\tau M = \operatorname{kern}\mathcal{N}(\phi)$.

If q is a path in Γ ending in $i \in \mathrm{V}\Gamma$, let $q^* \in \mathrm{D}(e_i\Lambda)$ be the dual element defined by

$$q^*(r) = \begin{cases} 1, & \text{if } q = r; \\ 0, & \text{otherwise} \end{cases}$$

for every path r ending in i.

Now let $B = \left\{\beta \in \mathrm{A}\Gamma \mid \mathrm{s}(\beta) \in \{1,\dots,n\}\right\} \setminus \{\alpha_1,\dots\alpha_{n-1}\}$ be the set of arrows leaving p. We have a minimal projective resolution

$$\bigoplus_{\beta\in B} \Lambda e_{\mathrm{t}(\beta)} \xrightarrow{\chi=(\chi_\beta)} \Lambda e_1 \longrightarrow U \longrightarrow 0$$

where the maps $\chi_\beta : \Lambda e_{\mathrm{t}(\beta)} \longrightarrow \Lambda e_1$ are right multiplications with the elements

$$u_\beta := \begin{cases} (\beta - \lambda_\beta \alpha_{\mathrm{t}(\beta)-1}\cdots\alpha_{\mathrm{s}(\beta)})\alpha_{\mathrm{s}(\beta)-1}\cdots\alpha_1, & \text{if } \mathrm{t}(\beta) \in \{1,\dots,n\}; \\ \beta\alpha_{\mathrm{s}(\beta)-1}\cdots\alpha_1, & \text{otherwise.} \end{cases}$$

The maps $\mathcal{N}(\chi_\beta) : \mathrm{D}(e_{\mathrm{t}(\beta)}\Lambda) \longrightarrow \mathrm{D}(e_1\Lambda)$ are then induced by left multiplication with the same elements, which implies that $\mathcal{N}(\chi_\beta)(e^*_{\mathrm{t}(\beta)})(q) = e^*_{\mathrm{t}(\beta)}(u_\beta q) = 0$ for every $\beta \in B$ and every path q ending in 1 (since the lengths of all paths appearing in $u_\beta q$ are greater than 0). We have thus found the K-linearly independent elements $(0,\ldots,e^*_{\mathrm{t}(\beta)},\ldots,0)$ in $\operatorname{kern}(\mathcal{N}(\chi))\cap\operatorname{soc}\bigoplus_{\beta\in B}\mathrm{D}(e_{\mathrm{t}(\beta)}\Lambda) = \operatorname{soc}\operatorname{kern}\mathcal{N}(\chi) = \operatorname{soc}\tau U$. Hence we need to have $\#B = 1$ if τU is to be uniserial.

We will assume from now on that $\#B = 1$, and we will denote the single element of B by β.
Case 1: $\mathrm{t}(\beta) \notin \{1,\ldots,n\}$.
This situation is rather easy; the following claim describes τU completely:
Claim: A K-basis of $\operatorname{kern}\mathcal{N}(\chi)$ is given by

$$\Big\{q^* \mid q \text{ a path ending in } \mathrm{t}(\beta) \text{ and not having } \beta\alpha_{\mathrm{s}(\beta)-1}\cdots\alpha_1 \text{ as left subpath}\Big\}.$$

Proof of the Claim: Using the bases $\{q^* \mid q \text{ a path ending in } 1\}$ of $\mathrm{D}(e_1\Lambda)$ and $\{(\beta\alpha_{\mathrm{s}(\beta)-1}\cdots\alpha_1 q)^* \mid q \text{ a path ending in } 1\} \cup \{q^* \mid q \text{ a path ending in } \mathrm{t}(\beta)$ and not having $\beta\alpha_{\mathrm{s}(\beta)-1}\cdots\alpha_1$ as a left subpath$\}$ of $\mathrm{D}(e_{\mathrm{t}(\beta)}\Lambda)$, the linear map $\mathcal{N}(\chi)$ is given by the matrix

$$\left(\begin{array}{ccc|c} 1 & & & \\ & \ddots & & 0 \\ & & 1 & \end{array}\right)$$

with kernel as claimed.

If we now assume that all paths ending in $\mathrm{t}(\beta)$ and not having $\beta\alpha_{\mathrm{s}(\beta)-1}\cdots\alpha_1$ as left subpath are comparable, we can denote by p' the longest such path. Then there is a unique Λ-isomorphism

$$U(p',(\mu_\alpha)) \longrightarrow \tau U = \operatorname{kern}\mathcal{N}(\chi)$$

which sends the image of $e_{\mathrm{s}(p')}$ in $U(p',(\mu_\alpha))$ to $(p')^*$.

On the other hand, if there are two non-comparable paths q_1, q_2 ending in $\mathrm{t}(\beta)$ but not having $\beta\alpha_{\mathrm{s}(\beta)-1}\cdots\alpha_1$ as left subpath, then the submodules of τU generated by q_1^* and q_2^* respectively are clearly not comparable.
Case 2: $\mathrm{t}(\beta) \in \{1,\ldots,n\}$.
Again, we describe τU first:
Claim: A K-basis of $\operatorname{kern}\mathcal{N}(\chi)$ is given by

$$\Big\{(\alpha_{\mathrm{t}(\beta)-1}\cdots\alpha_1 q)^* + \lambda_\beta(\beta\alpha_{\mathrm{s}(\beta)-1}\cdots\alpha_1 q)^* \mid q \text{ a path ending in } 1\Big\}$$
$$\cup\Big\{q^* \mid q \text{ a path ending in } \mathrm{t}(\beta) \text{ and having neither } \beta\alpha_{\mathrm{s}(\beta)-1}\cdots\alpha_1 \text{ nor } \alpha_{\mathrm{t}(\beta)-1}\cdots\alpha_1 \text{ as left subpaths}\Big\}.$$

Proof of the Claim: This time, we use the basis $\{(\beta\alpha_{\mathrm{s}(\beta)-1}\cdots\alpha_1 q)^* \mid q$ a path ending in $1\} \cup \{(\alpha_{\mathrm{t}(\beta)-1}\cdots\alpha_1 q)^* \mid q$ a path ending in $1\} \cup \{q^* \mid q$ a path ending in $\mathrm{t}(\beta)$ and having neither $\beta\alpha_{\mathrm{s}(\beta)-1}\cdots\alpha_1$ nor $\alpha_{\mathrm{t}(\beta)-1}\cdots\alpha_1$ as left subpaths} of $\mathrm{D}(e_{\mathrm{t}(\beta)}\Lambda)$; the map $\mathcal{N}(\chi)$ is then represented by the matrix

$$\left(\begin{array}{ccc|ccc|c} 1 & & & -\lambda & & & \\ & \ddots & & & \ddots & & 0 \\ & & 1 & & & -\lambda & \end{array}\right)$$

the kernel of which is as given above.

Let's assume now that all paths ending in $\mathrm{t}(\beta)$ and not having $\beta\alpha_{\mathrm{s}(\beta)-1}\cdots\alpha_1$ as left subpath are comparable; again we can denote by p' the longest such path. It is then possible to write $p' = \alpha_{\mathrm{t}(\beta)-1}\cdots\alpha_1 p''$ with some path p'' ending in 1. Then there is a unique Λ-isomorphism

$$U(p',(\mu_\alpha)) \longrightarrow \tau U = \operatorname{kern}\mathcal{N}(\chi)$$

sending the canonical top element of $U(p',(\mu_\alpha))$ to $(p')^* + \lambda_\beta(\beta\alpha_{\mathrm{s}(\beta)-1}\cdots\alpha_1 p'')^*$.

Conversely, if there are two non-comparable paths q_1, q_2 ending in $\mathrm{t}(\beta)$ but not having $\beta\alpha_{\mathrm{s}(\beta)-1}\cdots\alpha_1$ as left subpath, then we can define elements $w_s \in \tau U$, $s = 1, 2$, as follows:

$$w_s := \begin{cases} (\alpha_{\mathrm{t}(\beta)-1}\cdots\alpha_1 q_s')^* + \lambda_\beta(\beta\alpha_{\mathrm{s}(\beta)-1}\cdots\alpha_1 q_s')^*, \\ \qquad\qquad \text{if } q_s = \alpha_{\mathrm{t}(\beta)-1}\cdots\alpha_1 q_s' \text{ for some path } q_s'; \\ q_s^*, \quad \text{if } q_s \text{ does not have } \alpha_{\mathrm{t}(\beta)-1}\cdots\alpha_1 \text{ as left subpath.} \end{cases}$$

The submodules of τU generated by w_1 respectively w_2 are clearly not comparable. $\square$

For the convenience of the reader, we include here the dual version of Proposition 2.2.1.

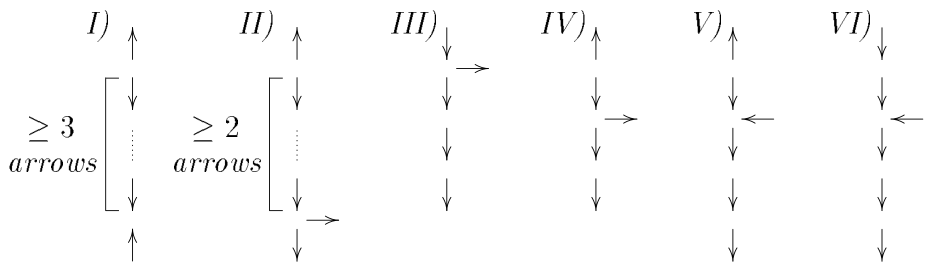
Proposition 2.2.2. *Let $p = 1 \xrightarrow{\alpha_1} 2 \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_{n-1}} n$ be a path in Γ and $\lambda_\alpha \in K$ for every $\alpha \wr p$. Write $U := U(p,(\lambda_\alpha))$. Then $\tau^{-1}U$ is defined and uniserial if and only if the following two conditions are both satisfied:*
(1) There is precisely one arrow γ entering p.
(2) Writing $\gamma : x \longrightarrow i$, then all paths starting in x and not having $\alpha_{n-1}\cdots\alpha_i\gamma$ as right subpath are comparable.

In this case, if p' is the longest path starting in x and not having $\alpha_{n-1}\cdots\alpha_i\gamma$ as right subpath, then

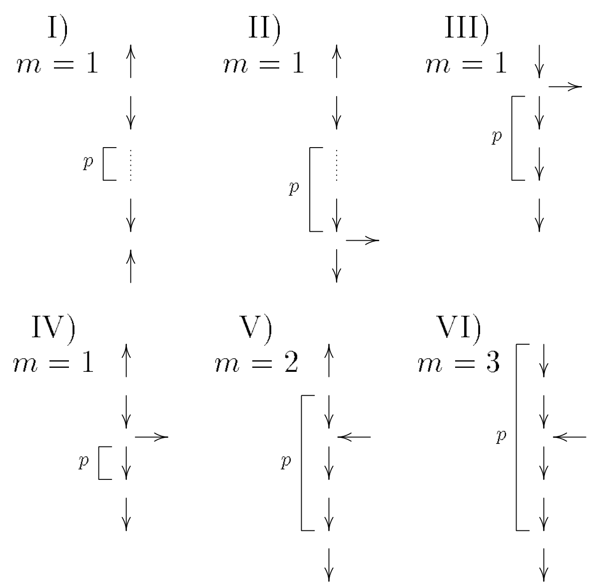
$$\tau^{-1}U = \begin{cases} U(p'), & \text{if } x \notin \{1,\dots,n\}; \\ U(p',(\mu_\alpha)), & \text{otherwise. Here, } \mu_\alpha = \begin{cases} 0, & \text{if } \gamma \neq \alpha \wr p'; \\ \lambda_\gamma, & \text{if } \gamma = \alpha. \end{cases} \end{cases}$$

In the initial examples we studied, it was a comparatively frequent phenomenon that every uniserial $U \in \Lambda\text{-}\mathrm{mod}$ had the property that either all the modules $\tau^i U$ for $i \geq 0$ were uniserial or undefined, or all the modules $\tau^{-i} U$ for $i \geq 0$ were uniserial or undefined, i. e. all uniserials were connected to projectives or injectives via a chain of τ-translations that only involved uniserials. Using Proposition 2.2.1 and its dual, we can completely pin down the hereditary algebras of finite type for which this is true:

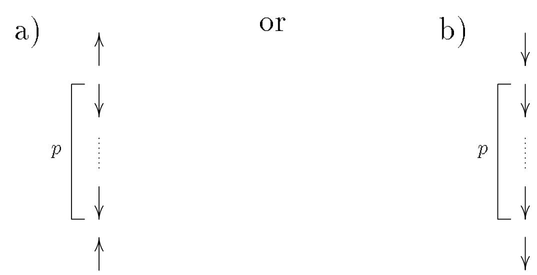
Corollary 2.2.3. *Let Γ be a quiver having one of the Dynkin diagrams $\mathbb{A}_n, \mathbb{D}_n, \mathbb{E}_6, \mathbb{E}_7, \mathbb{E}_8$ as underlying graph and write $\Lambda = K\Gamma$. Then there exist uniserial, non-projective and non-injective modules $U_1, \ldots, U_m \in \Lambda$-mod with $\tau U_i = U_{i-1}$ for $i \in \{2, \ldots, m\}$ and such that neither τU_1 nor $\tau^{-1} U_m$ are uniserial if and only if Γ contains one of the following quivers or their opposites as a subquiver:*



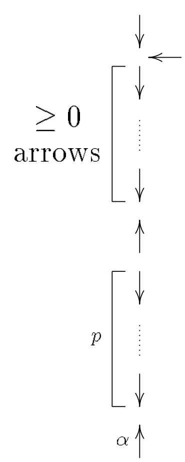
Proof. The above conditions are clearly true for Γ if and only if they are true for Γ^{op}. We will use this fact repeatedly in the sequel.
"$\Longleftarrow$": For each of the situations I to VI, we will exhibit a path p such that $U(p)$ plays the rôle of U_1 in the above condition. The value of m is also given.



"$\Longrightarrow$": Let p be the mast of U_1. Because of the structure of Γ, it follows that $U(p) \simeq U_1$.
Case 1. *For all vertices i on p, we have* $\mathrm{indeg}(i) + \mathrm{outdeg}(i) \leq 2$.
Since $U(p)$ is neither projective nor injective, Γ contains then either

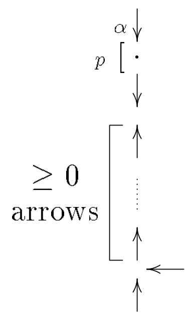


which leads to another case distinction:
Case 1.a) Since $\tau U(p)$ is not uniserial, Proposition 2.2.1 yields that in fact the following subquiver has to be present:



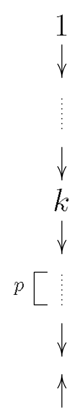
Let p_1 be the longest path in Γ starting in $\mathrm{s}(\alpha)$ and not containing α (note that because of the structure of the underlying graph of Γ, this specification is unambiguous). Let q_1 be the longest path terminating in $\mathrm{t}(p_1)$ and not being a left subpath of p_1. Then define p_2 to be the longest path starting in $\mathrm{s}(q_1)$ and not being a right subpath of q_1. Continue in this fashion until the end of the arm is reached. Now list the vertices of q_j in opposite order: $\mathrm{t}(q_j), i_1^{(j)}, \ldots, i_{l_j}^{(j)}, \mathrm{s}(q_j)$. By Proposition 2.2.2, we then have $\tau^{-1}U(p) = U(p_1)$, $\tau^{-2}U(p) = U(e_{i_1^{(1)}})$, $\tau^{-3}U(p) = U(e_{i_2^{(1)}})$, $\ldots$, $\tau^{-(l_1+1)}U(p) = U(e_{i_{l_1}^{(1)}})$, $\tau^{-(l_1+2)}U(p) = U(p_2)$, $\tau^{-(l_1+3)}U(p) = U(e_{i_1^{(2)}})$, $\ldots$, $\tau^{-(l_1+l_2+2)}U(p) = U(e_{i_{l_2}^{(2)}})$,

$\tau^{-(l_1+l_2+l_3)}U(p) = U(p_3)$ and so on. Eventually, an injective module is reached, and hence this case is impossible.
Case 1.b) If length$(p) = 0$, then Γ must contain the subquiver



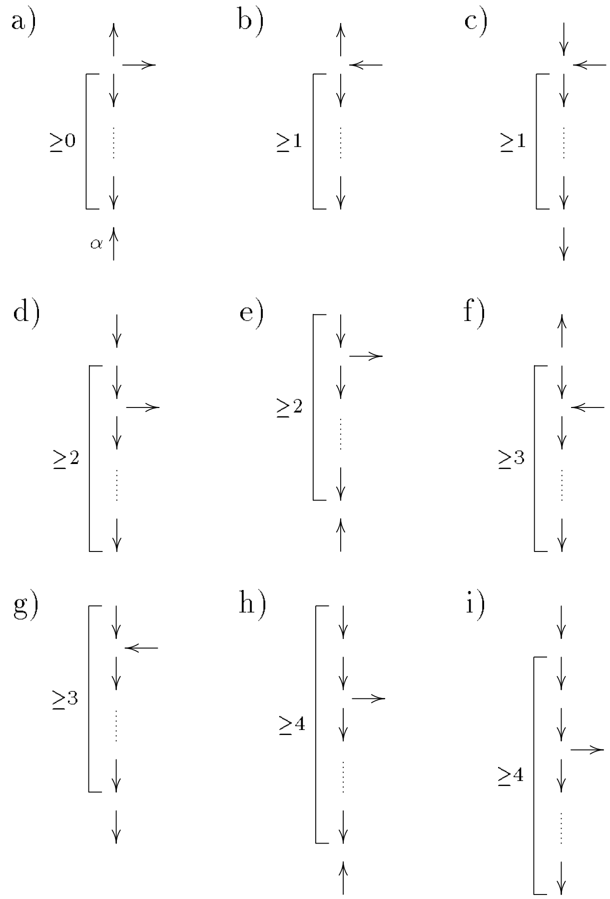
in order for $\tau U(p)$ not to be uniserial. This situation however has already been shown to be impossible in Case 1.a).

If length$(p) > 0$, again using the fact that $\tau U(p)$ is not uniserial, we can at least find a subquiver in Γ of the form

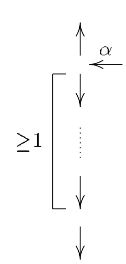


where $k \geq 1$ is as large as possible such that indeg(i) = outdeg$(i) = 1$ for $2 \leq i \leq k$. If outdeg$(1) \geq 2$, then we are in situation I) and are done. If indeg$(1) = 0$ and outdeg$(1) = 1$, then $\tau^{-k}U(p)$ is injective, and thus this cannot happen. The remaining case is indeg$(1) = 2$ and outdeg$(1) = 1$, which is covered by $\mathrm{II}^{\mathrm{op}}$.
Case 2. *There exists a vertex i on p with* indeg(i) + outdeg$(i) = 3$.
Since $U(p)$ is neither injective nor projective, we are left with the following (mutual exclusive) cases up to opposites. Again, we mark the path p and give

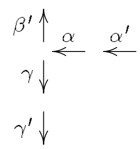
its length.



Case 2.a) As in Case 1.a), we define p_1 to be the longest path starting in $s(\alpha)$ and not containing α, and continue with q_1, p_2, q_2 and so on. Again, we see that $\tau^{-i}U(p)$ is uniserial or undefined for all $i \geq 0$, a contradiction.
Case 2.b) Since $\tau U(p)$ is not uniserial, Γ must in fact contain the subquiver

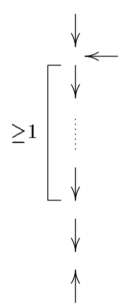


If $\mathrm{length}(p) \geq 2$, then we are in situation III; hence we will assume $\mathrm{length}(p) = 1$ from now on. The case $\mathrm{outdeg}(\mathrm{s}(\alpha)) = 2$ is covered by IV; if $\mathrm{outdeg}(\mathrm{s}(\alpha)) = 1$ and $\mathrm{indeg}(\mathrm{s}(\alpha)) = 1$, then $\tau^{-1}U(p) = U(\alpha)$ is injective. We are thus left with the situation



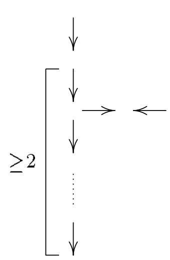
where $p = \gamma$. Then $\tau^{-1}U(p) = U(\beta'\alpha)$ and $\tau^{-2}U(p) = U(\gamma'\gamma\alpha\alpha')$. Since $\tau^{-2}U(p)$ is not injective, we have either $\mathrm{indeg}(\mathrm{s}(\alpha')) = 1$, in which case we are dealing with situation $\mathrm{VI}^{\mathrm{op}}$, or $\mathrm{indeg}(\mathrm{t}(\gamma')) = 1$, which is impossible since it develops like Case 1.a) above.

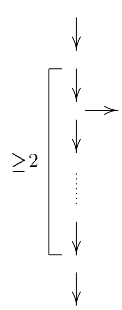
<u>Case 2.c)</u> Since $\tau(U(p))$ is supposed to be non-uniserial, the following subquiver



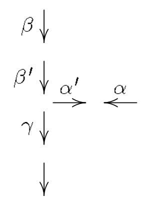
needs to be present. This however is situation $\mathrm{II}^{\mathrm{op}}$.

<u>Case 2.d)</u> Proposition 2.2.1 leaves two possibilities for non-uniserial $\tau U(p)$:

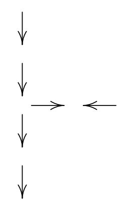
1) 

and
2) 

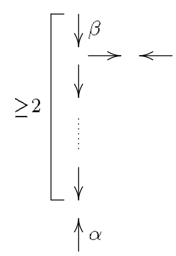
Case 2.d.1) We have length$(p) = 2$ because of the shape of Γ. The situation



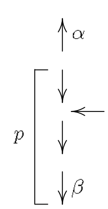
with $p = \gamma\beta'$ yields $\tau^{-1}U(p) = U(\alpha'\beta'\beta)$ which in turn develops as in Case 1.a) and is hence excluded.
Case 2.d.2) If length$(p) \geq 3$, then we are dealing with case III. Now assume length$(p) = 2$. Since $\tau^{-1}U(p)$ is not injective and because



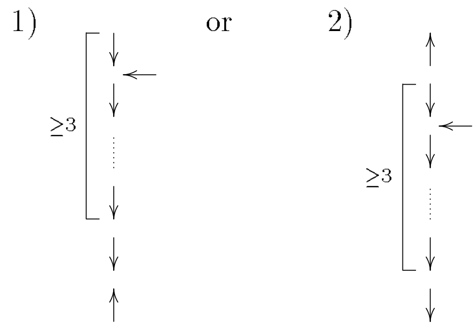
is not of finite type, we are left with the case $\mathrm{VI}^{\mathrm{op}}$.
Case 2.e) Since $\tau U(p)$ is not uniserial, we are either in situation IV or we deal with



The shape of Γ dictates that indeg$(\beta) = 0$, which implies that this situation develops again as in Case 1.a) and is hence impossible.
Case 2.f) If we are not in situation V, then length$(p) = 3$ and Γ has the subquiver



with $\mathrm{outdeg}(\beta) = 0$. If p_1 is the longest path in Γ terminating in $\mathrm{t}(\alpha)$ and not containing α, then $\tau U(p) = U(p_1)$ is uniserial, a contradiction.
Case 2.g) Since $\tau U(p)$ is not uniserial, we have either



Case 1) is covered by $\mathrm{II}^{\mathrm{op}}$ and case 2) by V.
Case 2.h) Again, $\tau U(p)$ being uniserial forces us to distinguish two subcases, but they both cannot appear in a quiver of finite type.
Case 2.i) is covered by $\mathrm{VI}^{\mathrm{op}}$. □

2.3. Triangles of Uniserials

In this section, we will exploit the results of Section 1.3 to describe the much more transparent hereditary situation. We will see that the maximal triangles of uniserials over hereditary algebras are disjoint and that virtually all irreducible maps between uniserial modules (more precisely: all but the radical embeddings of projectives and the socle factor projections of injectives) are in fact embeddable in such maximal triangles. An interesting consequence of this is the fact that, in the wild case, every regular uniserial is quasi-simple or the tip of a triangle of uniserials.

We start with strengthenings of the results from Section 1.3 for the hereditary situation.

Theorem 2.3.1. *Every triangle of uniserials over the hereditary algebra $\Lambda = K\Gamma$ is contained in a unique maximal one. In particular, different maximal triangles are disjoint.*

Proof. From the proof of Theorem 1.3.4, we know how the masts of the tips of the maximal triangles containing a given uniserial look like. In the absence of an ideal I of relations, it is clear that there is always precisely one such mast, giving rise to precisely one containing maximal triangle. □

Theorem 2.3.2. *Let $U \in \Lambda$-mod be a uniserial module over the hereditary algebra $\Lambda = K\Gamma$. If $p = 1 \xrightarrow{\alpha_1} 2 \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_{n-1}} n$ is a mast of U, then the*

following conditions are equivalent:
(a) *The radical embedding $JU \longrightarrow U$ is irreducible.*
(b) *We have* $n \geq 2$, $\operatorname{outdeg}(j) = 1$ *for* $j = 1, \dots, n-1$ *and either* $\operatorname{outdeg}(n) = 0$ *or* $\operatorname{indeg}(l) = 1$ *for* $l = 2, \dots, n$.

Proof. Taking into account the proven parts of Conjecture 1.2.1 and the fact that we don't have any relations here, it remains to be shown that, if there exists arrows $\beta' : n \longrightarrow y$ and $\gamma : z \longrightarrow i$ with $z \notin \{1, \dots, n\}$ and $i \in \{2, \dots, n\}$, then $JU \longrightarrow U$ is not irreducible.

Assuming that such arrows exist, we will construct a representation $V = ((V_x), (g_\alpha))$ and non-splitting Λ-linear maps $\Phi : JU \longrightarrow V$ and $\Psi : V \longrightarrow U$ such that $\Psi\Phi$ equals the radical embedding $JU \longrightarrow U$. As in the previous chapter, we identify U with the representation $U = ((U_x), (f_\alpha))$ of Γ, where

$$U_x = \begin{cases} K, & \text{if } x \in \{1, \dots, n\}; \\ 0, & \text{else} \end{cases}$$

and

$$f_{\alpha_i} = \mathrm{id} \quad \text{for every } i \in \{1, \dots, n-1\}.$$

In order to define V, set

$$V_x = \begin{cases} K, & \text{if } x \in \{1, y, z\}; \\ K \oplus K, & \text{if } x \in \{2, \dots, n\}; \\ 0, & \text{else} \end{cases}$$

$$g_\alpha = \begin{cases} \binom{0}{f_\alpha}, & \text{if } \mathrm{s}(\alpha) = 1 \text{ and } \mathrm{t}(\alpha) \in \{2, \dots, n\}; \\ f_\alpha \oplus f_\alpha, & \text{if } \{\mathrm{s}(\alpha), \mathrm{t}(\alpha)\} \subset \{2, \dots, n\}; \\ \binom{1}{-1}, & \text{if } \alpha = \gamma; \\ (0, 1), & \text{if } \alpha = \beta; \\ 0, & \text{else} \end{cases}$$

The homomorphisms

$$JU \xrightarrow{\Phi = (\Phi_x)} V \xrightarrow{\Psi = (\Psi_x)} U$$

are defined by

$$\Phi_x = \begin{cases} \binom{1}{0}, & \text{if } x \in \{2, \dots, n\}; \\ 0, & \text{else} \end{cases}$$

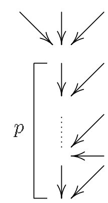
and

$$\Psi_x = \begin{cases} 1, & \text{if } x = 1; \\ (1,1), & \text{if } x \in \{2, \dots, n-1\}; \\ 0, & \text{else.} \end{cases}$$

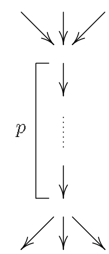
It is straightforward to check that these are indeed Λ-homomorphisms and that $\Psi\Phi$ equals the embedding $JU \longrightarrow U$. If $\chi = (\chi_x) : V \longrightarrow JU$ were a splitting of Φ, then we would have $\chi_1 = \chi_z = 0$, which would imply $\chi_i = 0$, but this is impossible. Every splitting $\chi = (\chi_x) : U \longrightarrow V$ of Ψ would satisfy $\chi_1(1) = 1$ and hence $\chi_n(1) = (0,1)$, implying $\chi_y \neq 0$, a contradiction. $\square$

A typical mast p of a uniserial module with irreducible radical embedding hence looks like one of the following:

Either i)



or ii)



Obviously, condition (i) corresponds to a projective module U; furthermore, condition (ii) is self-dual and inherited by all subpaths. These observations, combined with the dual version of the above theorem and with Theorem 1.3.4, already constitute the proof of the following corollary.

Corollary 2.3.3. *Let $U \in \Lambda\text{-}\mathrm{mod}$ be a uniserial module with length n.*
(a) *If the radical embedding $JU \longrightarrow U$ is irreducible, then there is a chain of irreducible maps*

$$J^{n-1}U \longrightarrow J^{n-2}U \longrightarrow \cdots \longrightarrow JU \longrightarrow U.$$

(b) *If the socle factor projection $U \longrightarrow U/\operatorname{soc} U$ is irreducible, then there is a chain of irreducible maps*

$$U \longrightarrow U/\operatorname{soc} U \longrightarrow \cdots \longrightarrow U/\operatorname{soc}^{(n-2)} U \longrightarrow U/\operatorname{soc}^{(n-1)} U.$$

(c) *The following are equivalent:*
(1) *U is the tip of a triangle of uniserials*
(2) *U is simple or both the radical embedding $JU \longrightarrow U$ and the socle factor projection $U \longrightarrow U/\operatorname{soc} U$ are irreducible maps.*

(3) *If* $p = 1 \xrightarrow{\alpha_1} 2 \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_{n-1}} n$ *is a mast of* U, *then* $\operatorname{outdeg}(j) = 1$ *for* $j = 1, \dots, n-1$ *and* $\operatorname{indeg}(l) = 1$ *for* $l = 2, \dots, n$.
(d) *If* U *is not projective and* $JU \longrightarrow U$ *is irreducible, then* U *is the tip of a triangle of uniserials.*
(e) *If* U *is not injective and* $U \longrightarrow U/\operatorname{soc} U$ *is irreducible, then* U *is the tip of a triangle of uniserials.*

Note that, in this corollary, we were able to characterize the tips without alluding to combinatorial terms such as masts or leaving arrows. This allows us to combine these results with the well-known information about the structure of Auslander-Reiten quivers of wild hereditary algebras contained for example in [**2**, Theorem 4.15]:

COROLLARY 2.3.4. *Let* $\Lambda = K\Gamma$ *be a wild hereditary algebra. Then all but finitely many uniserial modules in* Λ-mod *are regular and quasi-simple.*

PROOF. The dimension vectors of the preinjective (resp. postprojective) modules can be obtained through iterated application of the Coxeter transformation (resp. its inverse) to the dimension vectors of the indecomposable injectives (resp. projectives). In the wild case, the Coxeter matrix has exponential growth, and since the Loewy length of Λ puts an upper bound on the lengths of uniserial Λ-modules, almost all uniserials are regular. The regular components are isomorphic to $\mathbb{Z}\mathbb{A}_\infty$ and all irreducible maps are either injective or surjective. Combined with 1.1.3, this yields that both the radical embedding and the socle factor projection of the non-quasi-simple uniserials are irreducible. Using Corollary 2.3.3, we see then that these modules are actually tips of triangles of uniserials. On the other hand, we know that there can only exist finitely many such triangles. $\square$

Part 2

Coxeter Polynomials of Finite-Dimensional Algebras

CHAPTER 3

Methods to Determine Coxeter Polynomials

3.1. Introduction

The purpose of this chapter is to establish a reduction formula for the characteristic polynomial ϕ_Λ of the Coxeter matrix of a split finite-dimensional algebra Λ of finite global dimension. In fact, when Λ is put together from subalgebras in a certain natural fashion, we express ϕ_Λ in terms of the Coxeter polynomials of these subalgebras. In concrete computations, repeated application of this reduction principle offers a significant edge over direct use of the definition. This result will be presented in the following section; it has recently been published in [**5**].

Section 3.3 will then specialize on the hereditary case, where this reduction principle yields explicit expressions for the Coxeter polynomials of large families of quivers. Moreover, a combinatorial interpretation of the entries of the Coxeter matrices of hereditary path algebras allows us to establish formulas for Coxeter polynomials of some quivers which cannot be treated by the above-mentioned reduction process. These results were already presented in [**4**]; we include them again here in order to illustrate the method.

Throughout, Γ will be a finite quiver with vertex set $\mathrm{V}\Gamma$ and $\Lambda = K\Gamma/I$ will be a path algebra modulo an (admissible) ideal of relations over a field K such that $\dim_K \Lambda < \infty$. In order to avoid excessive subscripts, we will identify a vertex $e \in \mathrm{V}\Gamma$ with the corresponding primitive idempotent of Λ given by the path of length 0 centered at e. Recall that the $\mathrm{V}\Gamma \times \mathrm{V}\Gamma$ matrix

$$C_\Lambda = (\dim_K e\Lambda f)_{(e,f)\in \mathrm{V}\Gamma\times\mathrm{V}\Gamma}$$

is called the *Cartan matrix of* Λ and that, in case $|C_\Lambda| = \det(C_\Lambda) \neq 0$ (which is always satisfied if Λ has finite global dimension), the *Coxeter matrix of* Λ is defined as

$$\Phi_\Lambda := -{}^tC_\Lambda C_\Lambda^{-1},$$

where ${}^tC_\Lambda$ denotes the transpose of the matrix C_Λ. We will study the *Coxeter polynomial* $\phi_\Lambda(T) = |TE - \Phi_\Lambda|$ of Λ.

3.2. The Reduction Formula

Let r be a vertex of the quiver Γ and p a path in Γ. We say that p *properly passes through* r, if p can be written in the form $p = p_2 r p_1$ with paths p_1, p_2 in Γ of length ≥ 1.

For $n \in \mathbb{N}_0$, we say that *p properly passes through r precisely n times*, if p may be written in the form $p = p_{n+1} r p_n r \cdots r p_1$ with paths $p_1, \ldots, p_{n+1}$ of length ≥ 1 which do not properly pass through r.

Moreover, an admissible ideal I of relations in $K\Gamma$ is called *r-separated*, in case I can be generated, as an ideal, by a set R of relations such that for every $\sum_i \lambda_i p_i \in R$ with $\lambda_i \in K \setminus \{0\}$ and distinct paths p_i in Γ, none of the p_i properly passes through r.

We denote by $\Gamma \setminus \{r\}$ the quiver obtained from Γ by deleting the vertex r and all adjacent arrows. If Γ is the empty quiver without vertices and arrows, then $K\Gamma$ is defined to be the trivial zero-dimensional K-algebra with Coxeter polynomial 1.

The conclusion of the following lemma essentially allows us to count nonzero residue classes of paths in Λ in a similar way as one counts paths in the case of a finite-dimensional path algebra:

LEMMA 3.2.1. *Consider a finite-dimensional K-algebra $\Lambda = K\Gamma/I$ and let $r \in \mathrm{V}\Gamma$ be a vertex such that I is r-separated. Then*

(a) $$\dim_K r\Lambda r = 1$$

(b) *If we set $\check{\Gamma} := \Gamma \setminus \{r\}$ and $\check{\Lambda} := K\check{\Gamma}/(I \cap K\check{\Gamma})$, then the assignment*

$$u \otimes v \oplus w \longmapsto uv + w$$

yields an isomorphism

$$\Lambda r \underset{K}{\otimes} r\Lambda \;\oplus\; \check{\Lambda} \xrightarrow{\sim} \Lambda$$

of $\check{\Lambda}$-$\check{\Lambda}$-bimodules.

PROOF. For $n \in \mathbb{N}$, we denote by $P^{(n)}$ the K-subspace of $K\Gamma$ generated by all paths starting and ending in r and properly passing through r precisely $n-1$ times. Let $R \subset I$ be a generating set of relations which do not involve paths properly passing through r.

As an immediate consequence of the definitions, we get: If ρ is an element of R, and p is a path starting in r, while q is a path ending in r, then

$$q\rho p \in \bigcup_{n \geq 1} P^{(n)}. \tag{1}$$

Hence,

$$rIr = \bigoplus_{n \geq 1} I \cap P^{(n)}. \tag{2}$$

We write $\bar{P}^{(n)} := P^{(n)}/(I \cap P^{(n)})$. If moreover we denote by J the Jacobson radical of Λ, equation (2) yields

$$rJr = \bigoplus_{n\geq 1} \bar{P}^{(n)}. \tag{3}$$

Next, we prove

$$\bar{P}^{(n)} \simeq \bigotimes^n \bar{P}^{(1)},$$

where $\bigotimes^n \bar{P}^{(1)}$ is the n-fold tensor product of $\bar{P}^{(1)}$ with itself, taken over K. Together with $\dim_K \Lambda < \infty$ and (3), this will give us $\bar{P}^{(1)} = 0$ and hence statement (a).

The exact sequence

$$I \cap P^{(1)} \longrightarrow P^{(1)} \longrightarrow \bar{P}^{(1)} \longrightarrow 0$$

induces the upper sequence in the following commutative diagram, which has exact rows:

$$\begin{tikzcd}
\bigoplus_{k=1}^{n} \left(\bigotimes^{k-1} P^{(1)} \otimes (I \cap P^{(1)}) \otimes \bigotimes^{n-k} P^{(1)}\right) \arrow[r] \arrow[d, "f"] & \bigotimes^n P^{(1)} \arrow[r] \arrow[d, "g", "\sim"'] & \bigotimes^n \bar{P}^{(1)} \arrow[r] \arrow[d, "h"] & 0 \\
I \cap P^{(n)} \arrow[r] & P^{(n)} \arrow[r] & \bar{P}^{(n)} \arrow[r] & 0
\end{tikzcd}$$

Here, the maps f and g are defined by

$$g(p_1 \otimes p_2 \otimes \cdots \otimes p_n) = p_1 p_2 \cdots p_n$$

and

$$f(p_1 \otimes \cdots \otimes p_{k-1} \otimes x \otimes p_{k+1} \otimes \cdots \otimes p_n) = p_1 \cdots p_{k-1} x p_{k+1} \cdots p_n.$$

In order to prove that h is an isomorphism, it suffices to show that f is onto since g is an isomorphism. We only have to consider the case $n \geq 2$. Pick $x \in I \cap P^{(n)}$ and write $x = \sum_i \lambda_i q_i \rho_i p_i$ with $\lambda_i \in K \setminus \{0\}$, $\rho_i \in R$ and paths p_i, q_i starting resp. ending in r. Because of (1) and (2), we may assume $q_i \rho_i p_i \in P^{(n)}$ for all i. Write $p_i = p_{i2} r p_{i1}$ and $q_i = q_{i2} r q_{i1}$ where p_{i2} and q_{i1} have smallest possible length ≥ 0. Then we have $q_{i1}\rho_i p_{i2} \in I \cap P^{(1)}$. Moreover either $p_{i1} = r$ and $q_{i2} \in P^{(n-1)}$, or $p_{i1} \in P^{(n-1)}$ and $q_{i2} = r$, or else there is some $k_i \in \{2, \ldots, n-1\}$ such that $p_{i1} \in P^{(n-k_i)}$ and $q_{i2} \in P^{(k_i - 1)}$. In either case, we get $q_i \rho_i p_i \in \operatorname{im} f$ since multiplication of paths yields an isomorphism $\bigotimes^k P^{(1)} \xrightarrow{\sim} P^{(k)}$. Consequently, $x \in \operatorname{im} f$.

Now consider the map $\mu : \Lambda r \otimes_K r\Lambda \oplus \check{\Lambda} \to \Lambda$ from part (b) of the lemma. Obviously, it is well-defined, $\check{\Lambda}$-$\check{\Lambda}$-bilinear and surjective. In order to find a left inverse ν to μ, we start with a K-linear map $\nu_0 : K\Gamma \to \Lambda r \otimes_K r\Lambda \oplus \check{\Lambda}$, defined on the paths p in Γ as follows: if p can be written in the form $p = p_2 r p_1$ with paths p_1, p_2 (not necessarily of length ≥ 1), we set

$$\nu_0(p) := (p_2 + Ir) \otimes (p_1 + rI) \;\oplus\; 0.$$

This is well-defined, because if $p = q_2 r q_1$ is a different factorization of this kind, then either $q_1 = u q_1'$ or $q_2 = q_2' u$ with a suitable $u \in P^{(1)}$. But $P^{(1)} \subset I$ in view of (a), and thus $(q_2 + Ir) \otimes (q_1 + rI) = 0$. Analogously, one derives $(p_2 + Ir) \otimes (p_1 + rI) = 0$.
If p cannot be written in the form $p_2 r p_1$, we set

$$\nu_0(p) := 0 \oplus (p + I \cap K\check{\Gamma}).$$

Now suppose $x = \sum_i \lambda_i q_i \rho_i p_i \in I$ as above. If p_i can be written in the form $p_i = p_{i2} r p_{i1}$ with paths p_{i1} and p_{i2}, then $q_i \rho_i p_{i2} \in Ir$ and $\nu_0(q_i \rho_i p_i) = 0$. Similarly, if q_i admits a factorization $q_i = q_{i2} r q_{i1}$, we have $\nu_0(q_i \rho_i p_i) = 0$. The remaining case is $q_i \rho_i p_i \in I \cap K\check{\Gamma}$, and again we obtain $\nu_0(q_i \rho_i p_i) = 0$. Thus $\nu_0(x) = 0$ and ν_0 induces a K-linear map $\nu : \Lambda \to \Lambda r \otimes_K r\Lambda \;\oplus\; \check{\Lambda}$ which by construction is left inverse to μ. □

The *union* of quivers is given by the union of the vertex sets and the disjoint union of the arrow sets.

Now we are in a position to prove the main result:

THEOREM 3.2.2. *Let Γ_1, Γ_2 be two finite quivers with $\mathrm{V}\Gamma_1 \cap \mathrm{V}\Gamma_2 = \{r\}$, and let Γ be the union of Γ_1 and Γ_2. Suppose that $I \subset K\Gamma$ is an r-separated ideal of relations such that $\Lambda := K\Gamma/I$ is finite-dimensional. Set $\check{\Gamma}_1 := \Gamma_1 \setminus \{r\}$ and $\check{\Gamma}_2 := \Gamma_2 \setminus \{r\}$ and define the algebras $\Lambda_1, \check{\Lambda}_1, \Lambda_2, \check{\Lambda}_2$ canonically:*

$$\Lambda_i := K\Gamma_i/(I \cap K\Gamma_i) \quad \text{and} \quad \check{\Lambda}_i := K\check{\Gamma}_i/(I \cap K\check{\Gamma}_i) \quad \text{for } i = 1, 2.$$

Then

$$|C_{\Lambda_1}| = |C_{\check{\Lambda}_1}|, \quad |C_{\Lambda_2}| = |C_{\check{\Lambda}_2}| \quad \text{and} \quad |C_\Lambda| = |C_{\Lambda_1}||C_{\Lambda_2}|.$$

If this last determinant is nonzero, the Coxeter polynomial of Λ is

$$\phi_\Lambda = \phi_{\Lambda_1}\phi_{\check{\Lambda}_2} + \phi_{\check{\Lambda}_1}\phi_{\Lambda_2} - (T+1)\phi_{\check{\Lambda}_1}\phi_{\check{\Lambda}_2}.$$

PROOF. We need some additional notation: for every $e \in \mathrm{V}\check{\Lambda}_1$, let $a_e := \dim_K r\Lambda e$ and $\tilde{a}_e := \dim_K e\Lambda r$. Accordingly, for every $e \in \mathrm{V}\check{\Lambda}_2$, set $b_e := \dim_K r\Lambda e$ and $\tilde{b}_e := \dim_K e\Lambda r$. We consider a, $\tilde{a}$, b and $\tilde{b}$ as column vectors and write C, C_1, C_2, $\check{C}_1$, $\check{C}_2$ instead of C_Λ, C_{Λ_1}, C_{Λ_2}, $C_{\check{\Lambda}_1}$, $C_{\check{\Lambda}_2}$.

First we observe that $r\Lambda_1 e = r\Lambda e$ and $e\Lambda_1 r = e\Lambda r$ for all $e \in \mathrm{V}\Gamma_1$ since there are no arrows connecting $\mathrm{V}\check{\Gamma}_1$ and $\mathrm{V}\check{\Gamma}_2$ and $\dim_K r\Lambda r = 1$ by Lemma 3.2.1. Moreover, $I \cap K\Gamma_1$ is an r-separated ideal in $K\Gamma_1$ because every relation which does not involve any paths properly passing through r lies either in $K\Gamma_1$ or in $K\Gamma_2$. Applying Lemma 3.2.1 to Λ_1, we see that:

$$\dim_K r\Lambda_1 r = 1 \quad \text{and}$$
$$\dim_K e\Lambda_1 f = \dim_K e\check{\Lambda}_1 f + \tilde{a}_e a_f \quad \text{for all } e, f \in \mathrm{V}\check{\Gamma}_1.$$

Thus

$$C_1 = \left(\begin{array}{c|c} \check{C}_1 + \tilde{a}\,{}^t a & \tilde{a} \\ \hline {}^t a & 1 \end{array}\right),$$

and by subtracting suitable multiples of the last row from the others, we get $|C_1| = |\check{C}_1|$. Analogously, we obtain $|C_2| = |\check{C}_2|$. If we set $\check{\Gamma} := \Gamma \setminus \{r\}$ and $\check{\Lambda} := K\check{\Gamma}/(I \cap K\check{\Gamma})$, another application of Lemma 3.2.1 together with

$$C_{\check{\Lambda}} = \left(\begin{array}{c|c} \check{C}_1 & 0 \\ \hline 0 & \check{C}_2 \end{array}\right)$$

gives us

$$C = \left(\begin{array}{c|c|c} \check{C}_1 + \tilde{a}\,{}^t a & \tilde{a} & \tilde{a}\,{}^t b \\ \hline {}^t a & 1 & {}^t b \\ \hline \tilde{b}\,{}^t a & \tilde{b} & \check{C}_2 + \tilde{b}\,{}^t b \end{array}\right),$$

and hence $|C| = |\check{C}_1||\check{C}_2|$.

Now suppose C is invertible over $\mathbb{Q}$. Then the same is true for C_1, C_2, $\check{C}_1$ and $\check{C}_2$, and we write Φ, Φ_1, Φ_2, $\check{\Phi}_1$, $\check{\Phi}_2$ instead of Φ_Λ, Φ_{Λ_1}, Φ_{Λ_2}, $\Phi_{\check{\Lambda}_1}$, $\Phi_{\check{\Lambda}_2}$.

If A and B are invertible matrices such that $B = SA\,{}^t S$ for some invertible matrix S, we will write $A \sim B$. Note that in this case $S(-{}^t AA^{-1})S^{-1} = -{}^t BB^{-1}$, and therefore $-{}^t AA^{-1}$ and $-{}^t BB^{-1}$ have the same characteristic polynomial.

Obviously, we have

$$C_1 = \left(\begin{array}{c|c} \check{C}_1 + \tilde{a}\,{}^t a & \tilde{a} \\ \hline {}^t a & 1 \end{array}\right) \sim \left(\begin{array}{c|c} \check{C}_1 & \tilde{a} - a \\ \hline 0 & 1 \end{array}\right) =: D_1.$$

Moreover, observe that

$$D_1^{-1} = \left(\begin{array}{c|c} \check{C}_1^{-1} & \check{C}_1^{-1}(a - \tilde{a}) \\ \hline 0 & 1 \end{array}\right),$$

and hence

$$-{}^t D_1 D_1^{-1} = \left(\begin{array}{c|c} \check{\Phi}_1 & \check{\Phi}_1(a - \tilde{a}) \\ \hline {}^t(a - \tilde{a})\check{C}_1^{-1} & {}^t(a - \tilde{a})\check{C}_1^{-1}(a - \tilde{a}) - 1 \end{array}\right).$$

Similarly, Φ_2 and

$$\left(\begin{array}{c|c} {}^t(b-\tilde b)\check C_2^{-1}(b-\tilde b)-1 & {}^t(b-\tilde b)\check C_2^{-1} \\ \hline \check\Phi_2(b-\tilde b) & \check\Phi_2 \end{array}\right)$$

have the same characteristic polynomial. Applying the same reasoning to the full algebra Λ and using

$$\Phi_{\check\Lambda} = \left(\begin{array}{c|c} \check\Phi_1 & 0 \\ \hline 0 & \check\Phi_2 \end{array}\right),$$

we obtain that Φ and

$$\left(\begin{array}{c|c|c} \check\Phi_1 & \check\Phi_1(a-\tilde a) & 0 \\ \hline {}^t(a-\tilde a)\check C_1^{-1} & \lambda-1 & {}^t(b-\tilde b)\check C_2^{-1} \\ \hline 0 & \check\Phi_2(b-\tilde b) & \check\Phi_2 \end{array}\right)$$

have the same characteristic polynomial as well. Here, we set

$$\lambda = {}^t(a-\tilde a)\check C_1^{-1}(a-\tilde a) + {}^t(b-\tilde b)\check C_2^{-1}(b-\tilde b).$$

(Note that the quadratic form $\chi(x) = {}^t x C_{\check\Lambda}^{-1} x$ has significance in its own right because it is tightly connected to the Euler characteristic of the algebra $\check\Lambda$, see [**17**, p. 70].)

If finally we abbreviate

$$\begin{aligned} \alpha\ &:= -(T+1), \\ \alpha_1 &:= (T+1) - {}^t(a-\tilde a)\check C_1^{-1}(a-\tilde a), \\ \alpha_2 &:= (T+1) - {}^t(b-\tilde b)\check C_2^{-1}(b-\tilde b), \end{aligned}$$

we recognize the theorem as a consequence of the following lemma. □

LEMMA 3.2.3. *Let R be a commutative ring and $F \in \mathrm{M}_n(R)$ a matrix of the following form:*

$$F = \left(\begin{array}{c|c|c} F_1 & f_1 & 0 \\ \hline g_1 & \alpha_1+\alpha+\alpha_2 & g_2 \\ \hline 0 & f_2 & F_2 \end{array}\right)$$

where $F_1 \in \mathrm{M}_{n_1}(R)$, $F_2 \in \mathrm{M}_{n_2}(R)$, $n_1+n_2+1=n$, $\alpha,\alpha_1,\alpha_2 \in R$, $f_1, {}^tg_1 \in R^{n_1}$ and $f_2, {}^tg_2 \in R^{n_2}$. Then

$$|F| = \left|\begin{array}{c|c} F_1 & f_1 \\ \hline g_1 & \alpha_1 \end{array}\right| |F_2| + |F_1| \left|\begin{array}{c|c} \alpha_2 & g_2 \\ \hline f_2 & F_2 \end{array}\right| + \alpha|F_1||F_2|.$$

PROOF. Develop the determinant with respect to the (n_1+1)-th row or column. □

An obvious induction yields the following generalization of Theorem 3.2.2:

COROLLARY 3.2.4. *Let* Γ_i, $i = 1, \dots, m$, *be finite quivers with* $\mathrm{V}\Gamma_i \cap \mathrm{V}\Gamma_j = \{r\}$ *for* $i \neq j$, *and let* Γ *be the union of the* Γ_i. *Suppose that* $I \subset K\Gamma$ *is an* r*-separated ideal of relations such that* $\Lambda := K\Gamma/I$ *is finite-dimensional. Set* $\Lambda_i := K\Gamma_i/(I \cap K\Gamma_i)$ *and* $\check{\Gamma}_i := \Gamma_i \setminus \{r\}$ *and* $\check{\Lambda}_i := K\check{\Gamma}_i/(I \cap K\check{\Gamma}_i)$ *for* $i = 1, \dots, m$. *Then*

$$|C_{\Lambda_i}| = |C_{\check{\Lambda}_i}| \quad \textit{for all } i \quad \textit{and} \quad |C_\Lambda| = \prod_{i=1}^{m} |C_{\Lambda_i}|.$$

Moreover, if this last determinant is nonzero, we have

$$\phi_\Lambda = \Big(\prod_{i=1}^{m} \phi_{\check{\Lambda}_i}\Big)\Big(\sum_{i=1}^{m} \frac{\phi_{\Lambda_i}}{\phi_{\check{\Lambda}_i}} - (m-1)(T+1)\Big). \quad \square$$

By way of caution, we point out that the polynomials $\phi_{\check{\Lambda}_i}$ need not divide ϕ_Λ.

3.3. The Hereditary Case

If Λ is hereditary, i. e. if $\Lambda = K\Gamma$ and Γ is a finite quiver without oriented cycles, then the matrix C_Λ, and consequently also the matrix Φ_Λ and the polynomial ϕ_Λ, depend only on the quiver Γ and not on the base field K. In fact,

$$C_\Gamma := C_\Lambda = (\#\text{ paths from } f \text{ to } e \text{ in } \Gamma)_{(e,f)\in \mathrm{V}\Gamma\times\mathrm{V}\Gamma},$$

and the following matrix,

$$M_\Gamma := (\#\text{ arrows from } f \text{ to } e \text{ in } \Gamma)_{(e,f)\in \mathrm{V}\Gamma\times\mathrm{V}\Gamma},$$

satisfies $C_\Gamma^{-1} = E - M_\Gamma$, where E is the $\mathrm{V}\Gamma \times \mathrm{V}\Gamma$ identity matrix. With this in mind, one obtains a combinatorial interpretation of the entries of $\Phi_\Gamma := \Phi_\Lambda$ as follows. Namely, for $e, f \in \mathrm{V}\Gamma$, a *twisted path from* e *to* f is defined to be a tuple (β, p) with an arrow $\beta \in \mathrm{A}\Gamma$ starting in f and an oriented path p in Γ starting in e such that $\mathrm{t}(p) = \mathrm{t}(\beta)$. Roughly speaking, a twisted path consists of a 'standard' path to which we attach an inverted arrow at the end. With this convention, we obtain:

PROPOSITION 3.3.1. *The Coxeter matrix of a finite quiver* Γ *without oriented cycles is*

$$\Phi_\Gamma = (\#\textit{ twisted paths from } e \textit{ to } f - \#\textit{ paths from } e \textit{ to } f)_{(e,f)\in \mathrm{V}\Gamma\times\mathrm{V}\Gamma}.$$

PROOF. This follows from the above description of C_Γ and its inverse and the fact that the number of twisted paths from e to f is the sum of all products (number of paths from e to g)$\times$(number of arrows from f to g) for $g \in \mathrm{V}\Gamma$. $\square$

A more general formula for the entries of powers of Φ_Γ in terms of generalized twisted paths will be given in Section 4.2. What was called a "twisted path" here will be called a "1-endtwisted path" in that section.

It is interesting to note that the reduction formulas for the Coxeter polynomials and for the characteristic polynomials of the adjacency matrices of quivers of the type considered in Corollary 3.2.4 are exactly the same. (Of course, the term $(T+1)$, which is the Coxeter polynomial of a one-vertex quiver without arrows, has to be replaced by the characteristic polynomial of the corresponding adjacency matrix, i. e. by T.) The reason can again be found in Lemma 3.2.3.

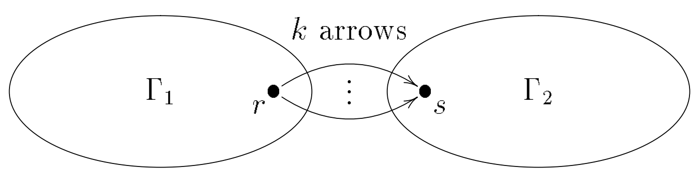
Observe moreover that there is a tight connection between $\phi_\Gamma := \phi_\Lambda$ and the characteristic polynomial of the adjacency matrix of Γ in case every vertex of Γ is either a sink or a source; see e. g. [**1**].

We set

$$v_k := \frac{T^k - 1}{T - 1} \quad \text{for every } k \in \mathbb{Z}.$$

The linear graph $\mathbb{A}_k$ with $k \geq 0$ vertices has Coxeter polynomial v_{k+1} as one easily derives from Theorem 3.2.2 by induction. The orientation of the arrows does not have any impact on the formula here; indeed, this is obviously true for $\mathbb{A}_2$, and thus follows inductively for higher values of k. In view of these remarks, a straightforward computation yields the following

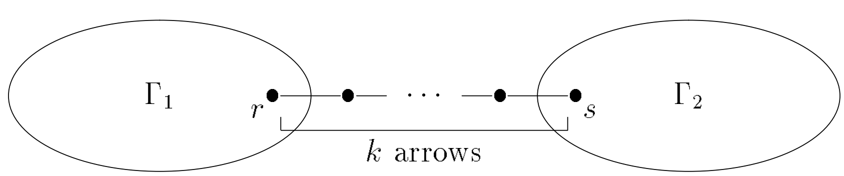
Corollary 3.3.2. *Let Γ_1 and Γ_2 be finite quivers without oriented cycles. Then the quiver*



has Coxeter polynomial

$$\phi_{\Gamma_1}\phi_{\Gamma_2} - k^2 T \phi_{\Gamma_1 \setminus \{r\}} \phi_{\Gamma_2 \setminus \{s\}}.$$

The quiver



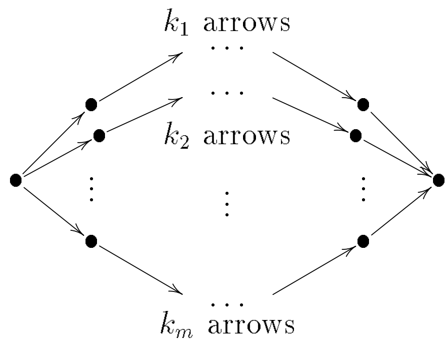
has Coxeter polynomial

$$v_k \phi_{\Gamma_1}\phi_{\Gamma_2} - T v_{k-1}\Big(\phi_{\Gamma_1 \setminus \{r\}}\phi_{\Gamma_2} + \phi_{\Gamma_1}\phi_{\Gamma_2 \setminus \{s\}}\Big) + T^2 v_{k-2}\phi_{\Gamma_1 \setminus \{r\}}\phi_{\Gamma_2 \setminus \{s\}},$$

irrespective of the orientation of the k arrows linking Γ_1 and Γ_2. □

We conclude this chapter with an example of a class of quivers which cannot be tackled with Theorem 3.2.2 and its corollaries:

PROPOSITION 3.3.3. *If Γ is the quiver*



with $m \in \mathbb{N}$ and $k_1, \ldots, k_m \in \mathbb{N}$ (the case $k_1 = \cdots = k_m = 1$ corresponding to an m-fold arrow between two vertices), then

$$\phi_\Gamma = \Big(\prod_{i=1}^{m} v_{k_i}\Big)\Big((m-1)^2(T+1)^2 - m^2T - (m-2)(T+1)\sum_{i=1}^{m}\frac{v_{k_i+1}}{v_{k_i}}\Big).$$

PROOF. We may assume $k_1 = \cdots = k_l = 1$ and $k_{l+1}, \ldots, k_m > 1$. For $i \in \{l+1, \ldots, m\}$, set

$$\Phi_i := \begin{pmatrix} & & & -1 \\ 1 & & & -1 \\ & \ddots & & \vdots \\ & & 1 & -1 \end{pmatrix} \in \mathrm{M}_{k_i-1}(\mathbb{Z})$$

and

$$\tilde{\Phi}_i := \begin{pmatrix} & & & 0 \\ 1 & & & 0 \\ & \ddots & & \vdots \\ & & 1 & 0 \end{pmatrix} \in \mathrm{M}_{k_i-1}(\mathbb{Z}).$$

(Entries which are not shown are assumed to be zero.) Then Φ_i is the Coxeter matrix of a linear graph with $k_i - 1$ vertices and all arrows pointing in the same direction. Counting the paths and twisted paths of Γ as in Proposition 3.3.1, we get

$$\Phi_\Gamma = \left(\begin{array}{cccccc}
lm+m-l-1 & (m-1) & (m-1) & & (m-1) & -m \\
\begin{matrix} l+1 \\ l \\ \vdots \\ l \end{matrix} & \boxed{\tilde{\Phi}_{l+1}} & \begin{matrix}1\\ \vdots \\ \vdots \\ 1\end{matrix} & & \begin{matrix}1\\ \vdots\\ \vdots\\ \vdots\end{matrix} & \begin{matrix}-1\\ \vdots\\ \vdots\\ \vdots\end{matrix} \\
\begin{matrix} l+1 \\ l \\ \vdots \\ l \end{matrix} & \begin{matrix}1\\ \vdots \\ \vdots \\ \vdots\end{matrix} & \boxed{\tilde{\Phi}_{l+2}} & & \vdots & \vdots \\
\vdots & \vdots & 1 & \ddots & 1 & \vdots \\
\begin{matrix} l+1 \\ l \\ \vdots \\ l \end{matrix} & \vdots & \vdots & & \boxed{\tilde{\Phi}_m} & \vdots \\
l & 1 & 1 & & 1 & -1
\end{array}\right)$$

Now consider the matrix $TE - \Phi_\Gamma$, add the last column to those corresponding to the last columns of the $\tilde{\Phi}_i$, and add the l-fold of the last column to the first. Next subtract the T-fold of the first row from the last. Finally develop the resulting determinant with respect to the last row and note that

$$\left|\begin{array}{ccccc}
T-m+l+1 & 1 & 1 & & 1 \\
-1 & \boxed{TE-\Phi_{l+1}} & & & \\
-1 & & \boxed{TE-\Phi_{l+2}} & & \\
 & & & \ddots & \\
-1 & & & & \boxed{TE-\Phi_m}
\end{array}\right|$$

is the Coxeter polynomial of a star all arrows of which point away from the center. By Corollary 3.2.4, it is equal to

$$\Big(\prod_{i=l+1}^{m} v_{k_i}\Big)\Big(\sum_{i=l+1}^{m} \frac{v_{k_i+1}}{v_{k_i}} - (m-l-1)(T+1)\Big).$$

When developing the remaining determinant

$$\begin{vmatrix} 1 & 1 & & 1 & m \\ \boxed{TE-\Phi_{l+1}} & & & & 1 \\ & \boxed{TE-\Phi_{l+2}} & & & \vdots \\ & & \ddots & & \vdots \\ & & & \boxed{TE-\Phi_{m}} & 1 \end{vmatrix}$$

with respect to the first row, it is crucial to observe that the determinant of the matrix obtained by replacing the last column of $TE - \Phi_i$ by ${}^t(1 \dots 1)$ is just v_{k_i-1}. To simplify the resulting expression, one uses the identity

$$(T+1) - \frac{v_{k+1}}{v_k} = T\frac{v_{k-1}}{v_k}.$$

The result follows. $\square$

CHAPTER 4

The Spectral Classes of Unicyclic Graphs

4.1. Introduction, Notation, and Preliminaries

In this chapter, we study the Coxeter polynomials of the different orientations of a finite graph that contains precisely one cycle. The collection of all those orientations that yield the same Coxeter polynomial is called a *spectral class* of that graph.

We will determine the number of spectral classes of such a unicyclic graph and show that their spectral radii are different in case the graph is wild. The proof uses the main result from the preceding chapter as well as covering techniques. Moreover, the entries of powers of the Coxeter matrix of an arbitrary finite quiver without oriented cycles are calculated.

The results presented here were obtained in joint work with Martha Takane and will appear in [**6**].

We consider a graph Δ and we will assume throughout that Δ is connected and does not have any loops (multiple edges between two vertices are allowed, however). We denote the set of vertices of the graph Δ by $\mathrm{V}\Delta$ and its edge set by $\mathrm{E}\Delta$. The graph is completely determined by its *adjacency matrix* $A_\Delta = (a_{ij}) \in \mathbb{Z}^{\mathrm{V}\Delta\times\mathrm{V}\Delta}$ which is the symmetric matrix whose ij-th entry is the number of edges in Δ between the vertices i and j.

If $<$ is a total order on $\mathrm{V}\Delta$, we write $(\Delta, <)$ for the following quiver: The set of vertices of $(\Delta, <)$ is the set of vertices of Δ, and there are a_{ij} arrows from i to j if $i > j$ and none otherwise. Note that $(\Delta, <)$, defined in this way, has no oriented cycles, and, furthermore, every quiver without oriented cycles having Δ as its underlying graph arises in this fashion from some ordering $<$. The Coxeter matrix $\Phi_{(\Delta,<)}$ (as defined in 3.1) and hence also the Coxeter polynomial $\phi_{(\Delta,<)}$ depend only on the quiver $(\Delta, <)$ and not on the specific choice of $<$. The spectral radius of $\Phi_{(\Delta,<)}$ will be denoted by $\rho_{(\Delta,<)}$. Recall that $\rho_{(\Delta,<)} = \max\{|\lambda| \mid \lambda \in \mathbb{C} \text{ is an eigenvalue of } \Phi_{(\Delta,<)}\}$.

If Δ' is a subgraph of Δ (i. e. Δ' is a graph having a subset of $\mathrm{V}\Delta$ as vertex set and a subset of $\mathrm{E}\Delta$ as edge set), then $<$ induces a total order on $\mathrm{V}\Delta'$, again denoted by $<$. The subgraph Δ' is said to be *full* if for any two vertices in Δ'

the set of edges between them is the same in Δ' as in Δ. We say that Δ' is a *proper subgraph* of Δ if it is a subgraph with $\Delta' \neq \Delta$.

An *essential cycle* of Δ is a full subgraph $\mathcal{C}$ of Δ having $m \geq 3$ vertices $x_0, \dots, x_{m-1}$ such that there are edges between x_i and x_{i+1} for $i = 0, \dots, m-2$ and also between x_{m-1} and x_0. The graph Δ is called *unicyclic* in case it contains precisely one essential cycle.

A vertex $x \in \mathrm{V}\Delta$ is called a *sink* of $(\Delta, <)$ if there is no arrow in $(\Delta, <)$ that starts in x; similarly, x is called a *source* if there is no arrow terminating in x. We say that $(\Delta, <)$ has *sink-source orientation* if every vertex is either a sink or a source. These quivers are also commonly called *bipartite*.

Now let $x \in \mathrm{V}\Delta$ be a source of $(\Delta, <)$. We denote by $r_x(\Delta, <)$ the quiver which is obtained from $(\Delta, <)$ by reversing the orientation of all the arrows containing x. In this way x becomes a sink of $r_x(\Delta, <)$.

We say that $r = r_{x_m} \cdots r_{x_1}$ is an *admissible change of orientation* for $(\Delta, <)$ provided that x_1 is a sources of $(\Delta, <)$ and x_i is a source of $r_{x_{i-1}} \cdots r_{x_1}(\Delta, <)$ for all i.

The following is a collection of well known results which we repeat here for the convenience of the reader.

PROPOSITION 4.1.1. *Let Δ be a finite graph and $<$ a total order on its vertex set.*
(i) *If $M = M_{(\Delta,<)} \in \mathbb{Z}^{\mathrm{V}\Delta \times \mathrm{V}\Delta}$ is the matrix whose ij-th entry is equal to the number of arrows from j to i in $(\Delta, <)$, then we have $A_\Delta = M + {}^tM$ and*

$$\Phi_{(\Delta,<)} = ({}^tM - E)^{-1}(E - M).$$

It follows that $\Phi_{(\Delta,<)^{\mathrm{op}}} = \Phi^{-1}_{(\Delta,<)}$ and also $\phi_{(\Delta,<)} = \phi_{(\Delta,<)^{\mathrm{op}}}$ because of $\Phi_{(\Delta,<)} = (E - M)^{-1}\,{}^t\Phi_{(\Delta,<)^{\mathrm{op}}}(E - M)$.
(ii) [**3**]: *$\phi_{(\Delta,<)} = \phi_{r(\Delta,<)}$ for every admissible change of orientation r of $(\Delta, <)$.*
(iii) [**18**]: *If $x \in \mathrm{V}\Delta$, then there exists an admissible change of orientation r of $(\Delta, <)$ such that x is the unique source of $r(\Delta, <)$.*
(iv) *Assume that Δ does not contain any essential cycles and let $<'$ be another total order on $\mathrm{V}\Delta$. Then there exists an admissible change of orientation of $(\Delta, <)$, say r, such that the quivers $r(\Delta, <)$ and $(\Delta, <')$ are equal. In particular, $\phi_{(\Delta,<)} = \phi_{(\Delta,<')}$ only depends on Δ and not on the orientation.*

Let Δ be finite and let $<$ be a total order on $\mathrm{V}\Delta$. It is well known that Δ is a Dynkin diagram if and only if $\rho_{(\Delta,<)} = 1$ and $\phi_{(\Delta,<)}(1) \neq 0$. Δ is an Euclidean graph if and only if $\rho_{(\Delta,<)} = 1$ and $\phi_{(\Delta,<)}(1) = 0$. We call both Δ and $(\Delta, <)$ *wild* in all other cases. The following theorem describes this situation.

THEOREM 4.1.2. *Let* $(\Delta, <)$ *be wild.*
(i) [**1, 18**]: *The spectral radius* $\rho_{(\Delta,<)}$ *is a simple root of* $\phi_{(\Delta,<)}$ *which is bigger than* 1. *Moreover,* $|\lambda| < \rho_{(\Delta,<)}$ *for all eigenvalues* $\lambda \neq \rho_{(\Delta,<)}$ *of* $\Phi_{(\Delta,<)}$. *In particular, if* $\mu \geq 0$ *and* $\phi_{(\Delta,<)}(\mu) < 0$, *then* $\mu < \rho_{(\Delta,<)}$.
(ii) [**1, 8**]: *Let* Δ' *be a proper subgraph of* Δ *(not necessarily full) and assume that either* $(\Delta, <)$ *has a sink-source orientation or else* Δ *has no essential cycle. Then* $\rho_{(\Delta',<)} < \rho_{(\Delta,<)}$.

4.2. Iterated Coxeter Transformations

Let Δ be a finite graph and $<$ a total order on $\mathrm{V}\Delta$. We are going to describe the entries of powers of the Coxeter matrix $\Phi = \Phi_{(\Delta,<)}$ in combinatorial terms, extending Proposition 3.3.1.

DEFINITION 4.2.1. A sequence $q = (p_\ell, \gamma_\ell, p_{\ell-1}, \gamma_{\ell-1}, \dots, \gamma_1, p_0)$ with $\ell \geq 0$, oriented paths $p_0, \dots, p_\ell$ and arrows $\gamma_1, \dots, \gamma_\ell$ in $(\Delta, <)$ is called an *ℓ-twisted path from* $\mathrm{s}(p_0)$ *to* $\mathrm{t}(p_\ell)$ if $\mathrm{t}(p_{i-1}) = \mathrm{t}(\gamma_i)$ and $\mathrm{s}(p_i) = \mathrm{s}(\gamma_i)$ for $i = 1, \dots, \ell$.

The sequence q is called *ℓ-endtwisted*, if in addition $0 = \mathrm{length}(p_\ell) := \#(\text{arrows belonging to } p_\ell)$ holds. If q is as above, define

$$\mathrm{length}_i(q) := \mathrm{length}(p_{i-1}) \text{ for } i = 1, \dots, \ell + 1$$

and

$$|q| := \ell + \sum_{i=0}^{\ell} \mathrm{length}(p_i).$$

The set of all ℓ-twisted paths in $(\Delta, <)$ from e to f is denoted by $T^\ell(e, f)$, and the subset of all ℓ-endtwisted paths from e to f is called $E^\ell(e, f)$.

PROPOSITION 4.2.2. *Let* n *be a natural number and* $e, f \in \mathrm{V}\Delta$. *Then*

$$(\Phi^n)_{e,f} = \sum_{\ell=1}^{n} (-1)^{n-\ell} \left(\sum_{q \in E^\ell(e,f)} \binom{|q| + n - \ell - 1}{n - \ell} - \sum_{q \in T^{\ell-1}(e,f)} \binom{|q| + n - \ell}{n - \ell} \right)$$

PROOF. If we set $M = (\#\text{arrows from } j \text{ to } i)_{i,j \in \mathrm{V}\Delta}$ and $C = (E - M)^{-1}$, then $\Phi = {}^tCM - {}^tC$, and an easy induction shows

$$\Phi^n = ({}^tCM - {}^tC)^n = \sum_{\ell=1}^{n} (-1)^{n-\ell} \Big(\sum_{\substack{(n_1, \dots, n_\ell) \\ n_i \geq 1 \text{ with } n = \sum_{i=1}^{\ell} n_i}} {}^tC^{n_1} \prod_{i=2}^{\ell} M\, {}^tC^{n_i} \Big) (M - E).$$

Now remember that C counts oriented paths in $(\Delta, <)$ and, more generally,

$$(C^k)_{e,f} = \sum_{p \text{ path from } f \text{ to } e} \binom{\text{length}(p) + k - 1}{k-1}.$$

Using the definition of $T^{\ell-1}(e,f)$, it follows

$$({}^tC^{n_1}(\prod_{i=2}^{\ell} M\,{}^tC^{n_i}))_{e,f} = \sum_{q \in T^{\ell-1}(e,f)} \prod_{i=1}^{\ell} \binom{\text{length}_{l+1-i}(q) + n_i - 1}{n_i - 1}$$

The result is now a consequence of the following identity, valid for all non negative $u, r_1, \ldots, r_\ell$:

$$\sum_{\substack{(u_1,\ldots,u_\ell) \\ u_i \geq 0 \text{ with } u = \sum_{i=1}^{\ell} u_i}} \prod_{i=1}^{\ell} \binom{r_i + u_i - 1}{u_i} = \binom{(\sum_{i=1}^{\ell} r_i) + u - 1}{u}$$

(Choose u elements with repetition from a disjoint union of ℓ sets, the i-th of which having r_i elements). $\square$

4.3. Galois Coverings

Let Δ, $\bar{\Delta}$ be (not necessarily finite) graphs and let $<$ and $\bar{<}$ be total orders on their respective vertex sets. Following [**11, 12**], we say that an epimorphism of quivers $\pi : (\bar{\Delta}, \bar{<}) \longrightarrow (\Delta, <)$ is a *Galois covering defined by the group* G, if the following conditions are satisfied:
1) $G \leq \text{Aut}((\bar{\Delta}, \bar{<}))$ is a group of quiver automorphisms which acts freely (i. e. the identity is the unique element of G leaving any vertex of $(\bar{\Delta}, \bar{<})$ fixed);
2) $\pi^{-1}(\pi x) = Gx$, for every arrow or vertex x in $(\bar{\Delta}, \bar{<})$.

Let $(\Delta^{(j)}, <)_{j \in \mathbb{N}}$ be a sequence of full finite subquivers of the (not necessarily finite) connected quiver $(\Delta, <)$. We assume that Δ has no essential cycles and is *bounded*, i. e. that there exists $K \in \mathbb{N}$ such that every $x \in \text{V}\Delta$ is contained in at most K edges. We say that $(\Delta^{(j)}, <)_j$ has *limit* $(\Delta, <)$ and write

$$(\Delta, <) = \lim_{j \to \infty} (\Delta^{(j)}, <)$$

if for any arrow α in $(\Delta, <)$, there exists $N \in \mathbb{N}$ such that α is an arrow in $(\Delta^{(j)}, <)$ for all $j \geq N$. In this case, according to [**9**] and [**10**], the sequence $(\rho_{(\Delta^{(j)},<)})$ converges, and we define

$$\rho_{(\Delta,<)} := \lim_{j \to \infty} \rho_{(\Delta^{(j)},<)}.$$

Obviously, this definition does not conflict with the previously defined $\rho_{(\Delta,<)}$ in case Δ is finite. Furthermore, $\rho_{(\Delta,<)}$ does not depend on the orientation $<$ by 4.1.1(iv).

In the sequel, we will constantly use the fact that, if $\pi : (\bar{\Delta}, \bar{<}) \longrightarrow (\Delta, <)$ is a Galois covering and Δ is bounded, then $\bar{\Delta}$ is bounded as well.

LEMMA 4.3.1. (i) *Let $(\Delta, <)$ be a finite, connected quiver and let $\pi : (\bar{\Delta}, \bar{<}) \longrightarrow (\Delta, <)$ be a Galois covering defined by the group G such that $\bar{\Delta}$ does not contain essential cycles. Then:*

i.1) $\rho_{(\bar{\Delta},\bar{<})} \leq \rho_{(\Delta,<)}$.

i.2) *If G is finite, then $\rho_{(\bar{\Delta},\bar{<})} = \rho_{(\Delta,<)}$.*

i.3) *If $(\Delta, <)$ is bipartite, Δ unicyclic, and $G = \mathbb{Z}$, then $\rho_{(\bar{\Delta},\bar{<})} = \rho_{(\Delta,<)}$.*

(ii) *Let $(\Delta, <)$ be a finite quiver whose underlying graph Δ is unicyclic. Then there exists a Galois covering $(\bar{\Delta}, \bar{<}) \longrightarrow (\Delta, <)$ with group $\mathbb{Z}$ such that $\bar{\Delta}$ does not contain essential cycles. Moreover, $\bar{\Delta}$ depends only on Δ and not on the orientation $<$.*

(iii) *Let Δ be a unicyclic graph with essential cycle $\mathcal{C}$ such that $|\mathrm{V}\mathcal{C}|$ is even. Then Δ admits a sink-source orientation $<_0$. Furthermore, whenever $<$ is another orientation of Δ, we have $\rho_{(\Delta,<_0)} \leq \rho_{(\Delta,<)}$.*

PROOF. (i): Parts (i.1) and (i.2) are contained in [**10**]; we derive (i.3) from [**9**] taking into account that $\mathbb{Z}$ is an amenable group.
(ii): Let $\mathcal{C} = \{x_0, \ldots, x_{m-1}\}$ be the essential cycle of $(\Delta, <)$. We define a Galois covering $\pi : (\bar{\Delta}, \bar{<}) \longrightarrow (\Delta, <)$ as follows: $\mathrm{V}\bar{\Delta} = \mathrm{V}\Delta \times \mathbb{Z}$, and the set of edges between the vertices (y, n), (z, m) is

$$\mathrm{E}\bar{\Delta}((y,n),(z,m)) = \begin{cases} \mathrm{E}\Delta(y,z), & \text{if } m = n \text{ and } \{y,z\} \neq \{x_0, x_{m-1}\}; \\ \mathrm{E}\Delta(x_{m-1}, x_0), & \text{if } m = n-1 \text{ and } y = x_{m-1}, z = x_0; \\ \varnothing, & \text{otherwise} \end{cases}$$

and $\bar{<}$ is the induced orientation of $\bar{\Delta}$. Then we set $\pi((y, n)) := y$ and extend this naturally to the arrows. The group $\mathbb{Z}$ operates in an obvious fashion on $(\bar{\Delta}, \bar{<})$.
(iii): It is clear that a sink-source orientation $<_0$ exists. If $<$ is another orientation, we use (ii) to find a graph $\bar{\Delta}$ and Galois coverings $\pi : (\bar{\Delta}, \bar{<}_0) \longrightarrow (\Delta, <_0)$ and $\pi' : (\bar{\Delta}, \bar{<}) \longrightarrow (\Delta, <)$ defined by $\mathbb{Z}$. The result follows:

$$\rho_{(\Delta,<_0)} \underset{(i.3)}{=} \rho_{(\bar{\Delta},\bar{<}_0)} = \rho_{(\bar{\Delta},\bar{<})} \underset{(i.1)}{\leq} \rho_{(\Delta,<)}.$$

□

PROPOSITION 4.3.2. *Let $(\Delta', <)$ be a proper (not necessarily full) subquiver of a wild unicyclic quiver $(\Delta, <)$. Assume Δ' contains no essential* *cycles. Then*

$$\rho_{(\Delta',<)} < \rho_{(\Delta,<)}.$$

PROOF. Let $\mathcal{C}$ be the essential cycle of Δ, with $m = |\mathrm{V}\mathcal{C}|$. By 4.3.1(i.2), we may assume without loss of generality that m is even. Thus, let $<_0$ be a sink-source orientation of Δ. Let $\pi : (\bar{\Delta}, \bar{<}) \to (\Delta, <)$ and $\pi_0 : (\bar{\Delta}, \bar{<}_0) \to (\Delta, <_0)$ be Galois coverings defined by $\mathbb{Z}$, as in 4.3.1(ii). We then get

$$\rho_{(\Delta',<)} \underset{4.1.1}{=} \rho_{(\Delta',<_0)} \underset{4.1.2}{<} \rho_{(\Delta,<_0)} \underset{4.3.1}{=} \rho_{(\bar{\Delta},\bar{<}_0)} = \rho_{(\bar{\Delta},\bar{<})} \underset{4.3.1}{\leq} \rho_{(\Delta,<)}.$$

$\square$

4.4. The Main Result

In this section, Δ will be a finite, unicyclic graph with essential cycle $\mathcal{C}$. We assume throughout that $\mathcal{C}$ has m vertices $x_0, \ldots, x_{m-1}$ and that there are edges between x_i and x_{i+1} for $i = 0, \ldots, m-2$ and also between x_{m-1} and x_0.

If $<$ is a total ordering on $\mathrm{V}\Delta$, we set

$$a_{(\Delta,<)} := \#\Big\{(u,v) \in \{(x_i, x_{i+1}) \mid 0 \leq i \leq m-2\} \cup \{(x_{m-1}, x_0)\} \Big| u > v\Big\}$$
$$b_{(\Delta,<)} := \#\Big\{(u,v) \in \{(x_i, x_{i+1}) \mid 0 \leq i \leq m-2\} \cup \{(x_{m-1}, x_0)\} \Big| u < v\Big\},$$

and define

$$v_{(\Delta,<)} := |a_{(\Delta,<)} - b_{(\Delta,<)}|.$$

Since $(\Delta, <)$ has no oriented cycles, both $a_{(\Delta,<)}$ and $b_{(\Delta,<)}$ are positive; furthermore, $v_{(\Delta,<)}$ clearly does not depend on the numbering of the vertices of $\mathcal{C}$. All three numbers depend only on the quiver $(\Delta, <)$ and not on the particular total order chosen. Loosely speaking, $a_{(\Delta,<)}$ counts the number of multi arrows in $\mathcal{C}$ pointing in clockwise direction, and $b_{(\Delta,<)}$ counts the others. If Δ is equal to the Euclidean diagram $\tilde{\mathbb{A}}_{m-1}$ and $a := a_{(\tilde{\mathbb{A}}_{m-1},<)}$, $b := b_{(\tilde{\mathbb{A}}_{m-1},<)}$, then we have

$$\phi_{(\tilde{\mathbb{A}}_{m-1},<)}(T) = (T^a - 1)(T^b - 1).$$

The following theorem is the main result of this chapter. The proof will follow after some preparations at the end of this section.

THEOREM 4.4.1. *Let Δ be a unicyclic graph whose essential cycle $\mathcal{C}$ has m vertices. Then:*
(i) *There exist integer polynomials $f, g \in \mathbb{Z}[T]$ (depending only on Δ), such that for every total order $<$ on $\mathrm{V}\Delta$:*

$$\phi_{(\Delta,<)} = f + g\phi_{(\mathcal{C},<)}.$$

Moreover, f and g are products of Coxeter polynomials of certain subgraphs of Δ having no essential cycles.
(ii) *The number of different spectral classes of Δ is equal to $[\frac{m}{2}]$ (=biggest integer less than or equal to $\frac{m}{2}$).*
(iii) *Let $<_1$ and $<_2$ be two total orders on $\mathrm{V}\Delta$. The following statements are equivalent:*
(a) $\phi_{(\Delta,<_1)} = \phi_{(\Delta,<_2)}$
(b) $v_{(\Delta,<_1)} = v_{(\Delta,<_2)}$
(c) *there exists an admissible change of orientation r of $(\Delta,<_1)$ such that $r(\Delta,<_1) = (\Delta,<_2)$ or $r(\Delta,<_1) = (\Delta,<_2)^{\mathrm{op}}$.*
Moreover, if Δ is wild, we have

$$\rho_{(\Delta,<_1)} < \rho_{(\Delta,<_2)} \Longleftrightarrow v_{(\Delta,<_1)} < v_{(\Delta,<_2)}.$$

Part (ii) of this theorem was proved in [**7**] in case $\Delta = \mathcal{C}$ is an essential cycle. We will eventually reduce the proof to this case, but our approach to the problem is different from Coleman's.

LEMMA 4.4.2. (i) *$a_{(\Delta,<)^{\mathrm{op}}} = m - a_{(\Delta,<)}$ and therefore $v_{(\Delta,<)} = v_{(\Delta,<)^{\mathrm{op}}}$.*
(ii) *Let $<_1$ and $<_2$ be orientations of Δ. We have $v_{(\Delta,<_1)} = v_{(\Delta,<_2)}$ if and only if there exists an admissible change of orientation r of $(\Delta,<_1)$ such that $r(\Delta,<_1) = (\Delta,<_2)$ or $r(\Delta,<_1) = (\Delta,<_2)^{\mathrm{op}}$. In this case, we have $\phi_{(\Delta,<_1)} = \phi_{(\Delta,<_2)}$.*

PROOF. (i) is clear.
(ii) "$\Leftarrow$" By (i) and induction, it is enough to take $r = r_x$, where $x \in \mathrm{V}\Delta$ is a source of $(\Delta,<_1)$, and show that $v_{r(\Delta,<_1)} = v_{(\Delta,<_1)}$. This is clear if $x \notin \mathrm{V}\mathcal{C}$ because the edges in $\mathcal{C}$ are not affected by the application of r. If, on the other hand, x is a vertex of $\mathcal{C}$, then the orientation of those edges of $\mathcal{C}$ that contain x will change, but the numbers $a_{(\Delta,<_1)}$, $b_{(\Delta,<_1)}$ and hence $v_{(\Delta,<_1)}$ remain the same.
"$\Rightarrow$" In the first case, we consider the situation when $a_{(\Delta,<_1)} = a_{(\Delta,<_2)}$. Pick $x \in \mathrm{V}\mathcal{C}$ arbitrary. We can find admissible changes of orientation s,t of $(\Delta,<_1)$ and $(\Delta,<_2)$ so that x is the unique source of both $s(\Delta,<_1)$ and $t(\Delta,<_2)$, and therefore of $\mathcal{C}$, according to 4.1.1(iii). We have $a_{s(\Delta,<_1)} = a_{t(\Delta,<_2)}$ and it is then clear that $s(\Delta,<_1)$ and $t(\Delta,<_2)$ must be the same quivers, which provides us with an admissible change of orientation r of $(\Delta,<_1)$ such that $r(\Delta,<_1) = (\Delta,<_2)$.
In the case $a_{(\Delta,<_1)} = b_{(\Delta,<_2)} = a_{(\Delta,<_2)^{\mathrm{op}}}$, using the same arguments, we can exhibit an admissible change of orientation r of $(\Delta,<_1)$ such that $r(\Delta,<_1) = (\Delta,<_2)^{\mathrm{op}}$. $\square$

PROPOSITION 4.4.3. *For $i = 1, \ldots, m-1$, let $(\mathcal{C}, <_i)$ be the orientation of $\mathcal{C}$ with unique source x_0 and $a_{(\mathcal{C},<_i)} = i$. We write $a_{ij} := a_{x_i x_j}$ for the number of edges between the vertex x_i and x_j and set $d := a_{01}a_{12}\cdots a_{(m-2)(m-1)}a_{(m-1)0}$ and $\phi_i := \phi_{(\mathcal{C},<_i)}$. Then*

$$\phi_i(T) - \phi_j(T) = d(T^j + T^{m-j} - T^i - T^{m-i}),$$

and, in particular, the ϕ_i are pairwise distinct for $1 \leq i \leq [\frac{m}{2}]$.

PROOF. We have $\phi_i(T) = \det(T(E - {}^t M_{(\Delta,<_i)}) + E - M_{(\Delta,<_i)})$, since $\det(E - {}^t M_{(\Delta,<_i)}) = 1$. Observe that $\left(T(E - {}^t M_{(\Delta,<_i)}) + E - M_{(\Delta,<_i)}\right)_{uv} = T+1$ if $u = v$, and $-[T(\#\text{arrows}(u \to v)) + (\#\text{arrows}(v \to u))]$ if $u \neq v$.

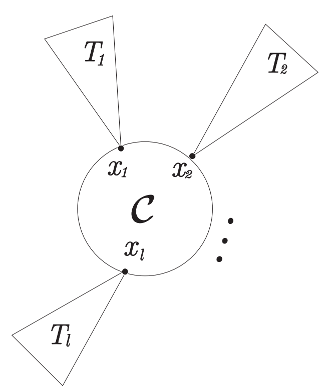
The Leibniz formula for the determinant then yields

$$\phi_i(T) = -d(T^i + T^{m-i}) + \sum_{\sigma \in M_m} \operatorname{sgn}(\sigma)(T+1)^{m-2\ell_\sigma} T^{\ell_\sigma} a^2_{i_1^{(\sigma)} j_1^{(\sigma)}} \cdots a^2_{i_{\ell_\sigma}^{(\sigma)} j_{\ell_\sigma}^{(\sigma)}}$$

where M_m is the set of all those permutations $\sigma \in S_m$ which can be written as a product of disjoint transpositions: $\sigma = (i_1^{(\sigma)} j_1^{(\sigma)}) \cdots (i_{l_\sigma}^{(\sigma)} j_{l_\sigma}^{(\sigma)})$. Clearly, the second half of this expression does not depend on i. $\square$

We have now assembled all ingredients for the proof of our main theorem:

PROOF OF THEOREM 4.4.1. (i) Since Δ is a unicyclic graph, it has the following shape:



where all T_i are trees and $x_i \in \mathrm{V}\mathcal{C} \cap \mathrm{V}T_i$, $i = 1, \ldots, \ell$. The statement follows then by induction on ℓ and Theorem 3.2.2, taking into account that the Coxeter polynomial of a graph without essential cycles does not depend on its orientation.

(ii) follows from (iii), below.

(iii) The equivalence of (b) and (c) was proved in Lemma 4.4.2(ii), while the implication (c)$\Rightarrow$(a) follows from 4.1.1(ii). To see (a)$\Rightarrow$(b), assume $v_{(\Delta,<_1)} \neq v_{(\Delta,<_2)}$. Using 4.1.1(iii), we find admissible changes of orientation r and s so that both $r(\mathcal{C},<_1)$ and $s(\mathcal{C},<_2)$ have unique source x_0. Because of $v_{r(\mathcal{C},<_1)} \neq v_{s(\mathcal{C},<_2)}$, Proposition 4.4.3 together with (i) shows that $\phi_{(\Delta,<_1)} \neq \phi_{(\Delta,<_2)}$.

Now assume Δ is wild, and $v_{(\Delta,<_1)} < v_{(\Delta,<_2)}$. Without loss of generality, we can assume that $a_{(\Delta,<_2)} \leq b_{(\Delta,<_2)}$. Since $v_{(\Delta,<_1)} < v_{(\Delta,<_2)}$, there exist real numbers $\alpha, \beta > 0$ such that $a_{(\Delta,<_1)} = a_{(\Delta,<_2)} + \alpha$, $b_{(\Delta,<_1)} = a_{(\Delta,<_2)} + \beta$, and thus $b_{(\Delta,<_2)} = a_{(\Delta,<_2)} + \alpha + \beta$. Write $\rho := \rho_{(\Delta,<_1)}$. We then get

$$\begin{aligned} \phi_{(\mathcal{C},<_2)}(\rho) - \phi_{(\mathcal{C},<_1)}(\rho) &\underset{4.4.3}{=} d(\rho^{a_{(\Delta,<_1)}} + \rho^{b_{(\Delta,<_1)}} - \rho^{a_{(\Delta,<_2)}} - \rho^{b_{(\Delta,<_2)}}) \\ &= -d\rho^{a_{(\Delta,<_2)}}(\rho^{\alpha} - 1)(\rho^{\beta} - 1) < 0 \end{aligned}$$

since $\rho > 1$ by 4.1.2. Note that $g(\rho) > 0$, where g is the polynomial from (i); this follows from 4.3.2 and 4.1.2(i). Using this, we get

$$\phi_{(\Delta,<_2)}(\rho) = \phi_{(\Delta,<_2)}(\rho) - \phi_{(\Delta,<_1)}(\rho) = g(\rho)(\phi_{(\mathcal{C},<_2)}(\rho) - \phi_{(\mathcal{C},<_1)}(\rho)) < 0,$$

which, again by 4.1.2(i), implies $\rho < \rho_{(\Delta,<_2)}$ and proves the theorem. $\square$

Bibliography

[1] Norbert A'Campo. Sur les valeurs propres de la transformation de Coxeter. *Invent.-Math.*, 33(1):61–67, 1976.

[2] Maurice Auslander, Idun Reiten, and Sverre Smalø. *Representation Theory of Artin Algebras*. Cambridge University Press, 1995.

[3] I. N. Bernstein, I. M. Gelfand, and V. A. Ponomarev. Coxeter functors and Gabriel's theorem. *Russian Math. Surveys*, 28, 1973.

[4] Axel Boldt. Coxetertransformationen. Diplomarbeit, Universität/Gesamthochschule Paderborn, Germany, 1992.

[5] Axel Boldt. Methods to determine Coxeter polynomials. *J. Lin. Alg. Appl.*, 230:151–164, 1995.

[6] Axel Boldt and Martha Takane. Some remarks about the spectral classes of unicyclic graphs. preprint, 1996.

[7] A. J. Coleman. Killing and the Coxeter transormation of Kac-Moody algebras. *Invent.-math*, 95:447–477, 1989.

[8] J. A. de la Peña and M. Takane. Spectral properties of Coxeter transformations and applications. *Arch. Math.*, 55:120–134, 1990.

[9] J. A. de la Peña and M. Takane. The spectral radius of the Galois covering of a finite graph. *J. Lin. Alg. Appl.*, 160:175–188, 1992.

[10] J. A. de la Peña and M. Takane. Some bounds for the spectral radius of a Coxeter transformation. *Tsukuba J. Math.*, 17(1):193–200, 1993.

[11] P. Dowbor and A. Skowroński. Galois coverings of representation-infinite algebras. *Comment. Math. Helvitici*, 62:311–337, 1987.

[12] P. Gabriel. The universal cover of a representation-finite algebra. In *Representations of algebras: proceedings of the third international conference on representations of algebras, held in Puebla, Mexico, August 4-8, 1980*, number 903 in Lecture Notes in Mathematics, pages 68–105. Springer-Verlag, 1981.

[13] F. R. Gantmacher. *The Theory of Matrices*, volume 1. Chelsea Publishing Company, 1959.

[14] Birge Huisgen Zimmermann. The geometry of uniserial representations of finite-dimensional algebras I. preprint, 1995.

[15] Birge Huisgen Zimmermann. The geometry of uniserial representations of finite-dimensional algebras III: finite uniserial type. accepted for publication in Trans. Am. Math. Soc., 1995.

[16] Frank Lukas. *Elementare Moduln über wilden erblichen Algebren*. PhD thesis, Universität/GH Düsseldorf, 1992.

[17] Claus Michael Ringel. *Tame algebras and integral quadratic forms.* Number 1099 in Lecture Notes in Mathematics. Springer-Verlag, 1984.

[18] Claus Michael Ringel. The spectral radius of the Coxeter transformations for a generalized Cartan matrix. *Math. Ann.*, 300, 1994.